

# Combinatorial embedded contact homology for toric contact manifolds

Keon Choi\*

## Abstract

Computing embedded contact homology (ECH) and related invariants of certain toric 3-manifolds (in the sense of Lerman [15]) has led to interesting new results in the study of symplectic embeddings [2, 4, 7]. Here, we give a combinatorial formulation of ECH chain complexes for general toric contact 3-manifolds. As a corollary, we prove Conjecture A.3 from [8].

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# 1 Introduction

Embedded contact homology (ECH) is an invariant of a contact 3-manifold. The goal of this paper is to combinatorially describe ECH chain complexes (ECC) of certain contact manifolds. A combinatorial formulation of Heegaard Floer homology (which is isomorphic to ECH by [3, 12]) is given in [16] but understanding ECC itself is useful for studying contact geometric properties lost under this isomorphism: e.g. usage of ECH capacities [2, 4, 7] and other obstructions [8] to symplectic embeddings.

In [11], Hutchings and Sullivan introduced “polygonal paths” and “rounding corners” to describe the generators and differentials of ECC for  $(T^3, \lambda_n)$  where  $\lambda_n := \cos(2\pi nx)dt_1 + \sin(2\pi nx)dt_1$ . We extend this result to all toric contact 3-manifolds, that is,  $(Y^3, \lambda)$  with a  $\lambda$ -preserving effective  $T^2$ -action. According to Lerman [15], such  $Y$  admits the contact moment map  $\mu_\lambda : Y \rightarrow (\mathfrak{t}^2)^*$  which factors through its orbital moment map  $a_\lambda : Y/T^2 \rightarrow (\mathfrak{t}^2)^*$ . If the action is free,  $Y/T^2$  is homeomorphic to  $\mathbb{R}/\mathbb{Z}$  and  $Y$  is diffeomorphic to  $T^3$ . Otherwise,  $Y/T^2$  is homeomorphic to  $[0, 1]$  and  $Y$  is diffeomorphic to a lens space (including  $S^1 \times S^2$ ). In either case,  $Y$  contains as a dense open submanifold a principal  $T^2$ -bundle  $Y^\circ := (0, 1) \times T^2$  and after re-identifying the fibres if necessary,  $\lambda|_{Y^\circ} = \pi^*(a_\lambda|_{(0,1)})$  where  $\pi : Y \rightarrow Y/T^2$ .

Let  $\mathbb{I} = [0, 1]$  and  $Y = \mathbb{I} \times T^2$  with the projection  $\pi_{\mathbb{I}} : Y \rightarrow \mathbb{I}$ . For any  $a : \mathbb{I} \rightarrow (\mathfrak{t}^2)^* = T^*T^2$ ,  $\pi_{\mathbb{I}}^*a : Y \rightarrow (\mathfrak{t}^2)^* \subset T^*Y$ , considered as a 1-form, is contact if and only if  $a \times a' > 0$  where  $\times$  is the standard cross product on  $(\mathfrak{t}^2)^* = \mathbb{R}^2$ . We call any such  $a$  an (abstract) orbital moment map. Our main theorem describes  $ECC(Y, \lambda, J)$  where  $\lambda$  is a certain perturbation of  $\pi_{\mathbb{I}}^*a$  for a generic orbital moment map  $a$  and  $J$  is a certain generic  $\lambda$ -admissible almost complex structure on  $\mathbb{R} \times Y$ . Recalling ECC is generated over  $\mathbb{Z}/2$  by admissible orbit sets of  $\lambda$  and the differential  $\partial$  counts ECH index 1  $J$ -holomorphic curves (see §2.1), we show:

**Theorem 1.1.** Let  $(Y, \lambda, J)$  be as above. For a pair  $(\alpha, \beta)$  of admissible orbit sets of  $\lambda$ ,  $\langle \partial\alpha, \beta \rangle = 1 \in \mathbb{Z}/2$  if and only if the region  $\mathcal{R}_{\alpha, \beta}$  associated to it can be written as a concatenation  $\mathcal{T}_1 \mathcal{R}' \mathcal{T}_2$  where  $\mathcal{T}_i$  are trivial regions and  $\mathcal{R}'$  is a non-local, indecomposable,  $a$ -positive, minimally positive and almost minimally decorated region.

Before giving precise definitions, it is useful to have in mind:

- (a) The (rough) correspondence between: a trivial region and a trivial cylinder; an indecomposable region and an irreducible  $J$ -holomorphic curve; a concatenation of regions and a disjoint union of  $J$ -holomorphic curves; and  $a$ -positivity and intersection positivity.
- (b) The combinatorial ECH index of a non-local, indecomposable,  $a$ -positive region is non-negative. It is zero if and only if the region is minimally positive and minimally decorated.

Since the Reeb vector field  $\bar{R}$  of  $\pi_{\mathbb{I}}^*a$  at  $(x, t_1, t_2) \in Y$  takes values in  $\ker a'(x) \subset \mathfrak{t}^2 = T_{(x, t_1, t_2)}^{vert} Y$ , an  $S^1$ -family  $\bar{\rho}_x$  of embedded orbits foliates  $\{x\} \times T^2$  whenever  $a'(x)$  is a multiple of an integral vector. The set of such  $x$  is dense generically but a technical argument allows us to only consider  $\bar{\rho}_x$  containing orbits whose action is less than a fixed constant  $L$  (See §2.1.) Then, following Bourgeois [1], we perturb  $\pi_{\mathbb{I}}^*a$  to  $\lambda$ , which has exactly two embedded orbits with action less than  $L$  (one elliptic orbit  $e_x$  and one positive hyperbolic orbit  $h_x$ ) for each such  $\bar{\rho}_x$  (see §2.2). In addition to  $[e_x] = [h_x] \in H_1(T^2)$ ,  $\bar{\rho}_x$  has another important attribute:

**Definition 1.2.** We say  $a$  is *convex* at  $x$  (or  $\bar{\rho}_x$  is *convex*) if  $a'(x) \times a''(x) > 0$ ;  $a$  is *concave* at  $x$  (or  $\bar{\rho}_x$  is *concave*) if  $a'(x) \times a''(x) < 0$ .

In  $(T^3, \lambda_n)$ ,  $a_{\lambda_n}(x) = (\cos 2\pi nx, \sin 2\pi nx)$  so every  $\bar{\rho}_x$  is convex whereas in  $(S^3, \lambda_{std})$ ,  $a_{\lambda_{std}}(x) = (1 - x, x)$  so no  $\bar{\rho}_x$  is convex or concave. By genericity of  $a$ , we assume every orbit of action less than  $L$  is either convex or concave.

**Definition 1.3.** Let  $\Lambda \subset \mathfrak{t}^2$  be the kernel of the exponential map (hence, naturally identified with  $H_1(T^2)$ ). A *(lattice) path*  $\bar{\mathcal{P}}$  is a function

$$(v_{\bar{\mathcal{P}}}, c_{\bar{\mathcal{P}}}, m_{\bar{\mathcal{P}}}) : \mathbb{I} \rightarrow \bar{\mathcal{V}} := \Lambda \times \{\pm 1, 0\} \times \mathbb{N}$$

such that  $v_{\bar{\mathcal{P}}}, c_{\bar{\mathcal{P}}}$  and  $m_{\bar{\mathcal{P}}}$  (read *edge*, *convexity* and *multiplicity*) are non-zero on the same finite set  $\text{supp } \bar{\mathcal{P}}$  (read the *support* of  $\bar{\mathcal{P}}$ ) and  $v_{\bar{\mathcal{P}}}(x)$  is primitive whenever non-zero. We write  $m(\bar{\mathcal{P}}) = \sum_x m_{\bar{\mathcal{P}}}(x)$  and  $[\bar{\mathcal{P}}] := \sum_x m_{\bar{\mathcal{P}}}(x) \cdot v_{\bar{\mathcal{P}}}(x)$  and say:

- (a) Two paths  $\bar{\mathcal{P}}_1$  and  $\bar{\mathcal{P}}_2$  are *compatible* if  $v_{\bar{\mathcal{P}}_1}(x) = v_{\bar{\mathcal{P}}_2}(x)$  and  $c_{\bar{\mathcal{P}}_1}(x) = c_{\bar{\mathcal{P}}_2}(x)$  for every  $x \in \text{supp}(\bar{\mathcal{P}}_1) \cap \text{supp}(\bar{\mathcal{P}}_2)$ . In this case, their *union*  $\bar{\mathcal{P}}_1 \cup \bar{\mathcal{P}}_2$  is the path with  $m_{\bar{\mathcal{P}}_1 \cup \bar{\mathcal{P}}_2} = m_{\bar{\mathcal{P}}_1} + m_{\bar{\mathcal{P}}_2}$  and compatible with each  $\bar{\mathcal{P}}_i$ .

A *decoration*  $\mathcal{P}$  of  $\bar{\mathcal{P}}$  is a function  $(v_{\mathcal{P}}, c_{\mathcal{P}}, m_{\mathcal{P}}^e, m_{\mathcal{P}}^h) : \mathbb{I} \rightarrow \mathcal{V} := \Lambda \times \{\pm 1, 0\} \times \mathbb{N}^2$  ( $m_{\mathcal{P}}^e$  and  $m_{\mathcal{P}}^h$  read *elliptic* and *hyperbolic multiplicity*) with  $\bar{\mathcal{P}} = (v_{\mathcal{P}}, c_{\mathcal{P}}, m_{\mathcal{P}}^e + m_{\mathcal{P}}^h)$ .

We use the term “path” because we can depict  $\bar{\mathcal{P}}$  as a piecewise linear curve in  $\mathfrak{t}^2$  by concatenating  $v_{\bar{\mathcal{P}}}(x)$  with multiplicity  $m_{\bar{\mathcal{P}}}(x)$  in order of increasing  $x$ , where each instance of  $v_{\bar{\mathcal{P}}}(x)$  is labelled with  $\check{x}$  if  $c_{\bar{\mathcal{P}}}(x) = 1$  (*convex*) and with  $\hat{x}$  if  $c_{\bar{\mathcal{P}}}(x) = -1$  (*concave*). For a decorated path  $\mathcal{P}$ , we label each edge with  $\check{e}_x, \check{h}_x, \hat{h}_x$  or  $\hat{e}_x$  (this is unique only up to shuffling  $e/h$  labels at the same  $x$ ) as we see shortly in Figure 1.

**Definition 1.4.** A *(lattice) region*  $\bar{\mathcal{R}}$  is a pair  $(\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^1)$  of compatible lattice paths with  $[\bar{\mathcal{P}}^0] = [\bar{\mathcal{P}}^1]$ . We write  $c_{\bar{\mathcal{R}}} = c_{\bar{\mathcal{P}}^0 \cup \bar{\mathcal{P}}^1}$ ,  $v_{\bar{\mathcal{R}}} = v_{\bar{\mathcal{P}}^0 \cup \bar{\mathcal{P}}^1}$ ,  $m_{\bar{\mathcal{R}}} = m_{\bar{\mathcal{P}}^0 \cup \bar{\mathcal{P}}^1}$ , and  $m(\bar{\mathcal{R}}) = \sum_x m_{\bar{\mathcal{R}}}(x)$ . The *slice class* of  $\bar{\mathcal{R}}$  at  $x_0 \in \mathbb{I}$  is

$$\sigma_{\bar{\mathcal{R}}}(x_0) := - \sum_{x < x_0} m_{\bar{\mathcal{P}}^0}(x) \cdot v_{\bar{\mathcal{P}}^0}(x) + \sum_{x < x_0} m_{\bar{\mathcal{P}}^1}(x) \cdot v_{\bar{\mathcal{P}}^1}(x) \in \Lambda$$

and the *support* of  $\bar{\mathcal{R}}$  is  $\text{supp}(\bar{\mathcal{R}}) := \text{supp}(m_{\bar{\mathcal{R}}}) \cup \text{supp}(\sigma_{\bar{\mathcal{R}}})$ . We say:

- (a)  $\bar{\mathcal{R}}$  is *local* if  $\bar{\mathcal{P}}^0 = \bar{\mathcal{P}}^1$  and *empty* if  $\bar{\mathcal{P}}^0 = \bar{\mathcal{P}}^1 = 0$ .
- (b) Two regions  $\bar{\mathcal{R}}_1 = (\bar{\mathcal{P}}_1^0, \bar{\mathcal{P}}_1^1)$  and  $\bar{\mathcal{R}}_2 = (\bar{\mathcal{P}}_2^0, \bar{\mathcal{P}}_2^1)$  are *composable* at  $x_0$  if  $\bar{\mathcal{P}}_1^0, \bar{\mathcal{P}}_1^1, \bar{\mathcal{P}}_2^0$  and  $\bar{\mathcal{P}}_2^1$  are pairwise compatible and  $\max(\text{supp } \bar{\mathcal{R}}_1) \leq x_0 \leq \min(\text{supp } \bar{\mathcal{R}}_2)$ . In this case, their *concatenation*  $\bar{\mathcal{R}}_1 \bar{\mathcal{R}}_2$  is  $(\bar{\mathcal{P}}_1^0 \cup \bar{\mathcal{P}}_2^0, \bar{\mathcal{P}}_1^1 \cup \bar{\mathcal{P}}_2^1)$  and  $\bar{\mathcal{R}}_1 \bar{\mathcal{R}}_2$  is said to *decompose* at  $x_0$ . We say  $\bar{\mathcal{R}}$  is *indecomposable* if it cannot be written as a concatenation of two non-empty regions. Any  $\bar{\mathcal{R}}$  can be uniquely written as a concatenation  $\bar{\mathcal{R}}_1 \cdots \bar{\mathcal{R}}_d$  where each  $\bar{\mathcal{R}}_i$ , called a *factor*, is non-empty and indecomposable.
- (c) A non-local indecomposable region  $\bar{\mathcal{R}} = (\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^1)$  is *positive* if, for each  $i$ ,  $v_{\bar{\mathcal{P}}^i}(x) \times \sigma_{\bar{\mathcal{R}}}(x) \geq 0$  with equality only if  $c_{\bar{\mathcal{P}}^i}(x) \neq (-1)^{i+1}$ . It is *minimally positive* if it is positive, each non-zero  $\sigma_{\bar{\mathcal{R}}}(x)$  is primitive and, for each  $i$ ,  $v_{\bar{\mathcal{P}}^i}(x) \times \sigma_{\bar{\mathcal{R}}}(x) \leq 1$  with equality only if  $c_{\bar{\mathcal{P}}^i}(x) \neq (-1)^i$ . A general region  $\bar{\mathcal{R}}$  is (*minimally*) *positive* if each of its non-local factors is.

A *decoration*  $\mathcal{R}$  of  $\bar{\mathcal{R}} = (\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^1)$  is a pair  $(\mathcal{P}^0, \mathcal{P}^1)$  of decorations  $\mathcal{P}^i$  of  $\bar{\mathcal{P}}^i$ . We say:

- (d)  $\mathcal{R}$  is *trivial* if  $\mathcal{P}^0 = \mathcal{P}^1$ .
- (e)  $\mathcal{R}$  is *minimally decorated* if, for each  $i$ ,  $m_{\mathcal{P}^i}^e(x) = 0$  whenever  $c_{\mathcal{P}^i}(x) = (-1)^i$  and  $m_{\mathcal{P}^i}^h(x) = 0$  whenever  $c_{\mathcal{P}^i}(x) = (-1)^{i+1}$ . It is *almost minimally decorated* if  $\sum_x |m_{\mathcal{P}^0}^e(x) - m_{\mathcal{P}_{\min}^0}^e(x)| + |m_{\mathcal{P}^1}^e(x) - m_{\mathcal{P}_{\min}^1}^e(x)| = 1$  where  $(\mathcal{P}_{\min}^0, \mathcal{P}_{\min}^1)$  is the minimal decoration of  $\bar{\mathcal{R}}$ .

A decorated path  $\mathcal{P}$  and a decorated region  $\mathcal{R}$  inherit terminologies and operations of their underlying undecorated  $\bar{\mathcal{P}}$  and  $\bar{\mathcal{R}}$ . One caveat is that a decomposition of  $\bar{\mathcal{R}}$  at  $x_0$  is unique only up to re-distributing elliptic/hyperbolic multiplicities at  $x_0$ . We depict  $\bar{\mathcal{R}} = (\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^1)$  by a closed (not necessarily embedded) polygon between  $\bar{\mathcal{P}}^0$  and  $\bar{\mathcal{P}}^1$ .

In Figure 1(b), three lattice regions are drawn with  $\mathcal{P}^0, \mathcal{P}^1$  and  $\sigma_{\mathcal{R}}$  in red, blue and dotted arrows, respectively. They are minimally positive: each triangle formed by  $v_{\mathcal{P}^i}(x)$  and  $\sigma_{\mathcal{R}}(x)$  is either degenerate or primitive with the right orientation; and the convexity  $c_{\mathcal{P}^i}$  satisfies the requirement, e.g. for the third region,  $v_{\mathcal{P}^1}(x_2)$  is not parallel to  $\sigma_{\mathcal{R}}(x_2)$  so  $c_{\mathcal{P}^1}(x_2) = 1$  while  $v_{\mathcal{P}^1}(x_4)$  is parallel to  $\sigma_{\mathcal{R}}(x_4)$  so  $c_{\mathcal{P}^1}(x_4) = -1$ , and so on. They are also almost minimally decorated: we recover the minimal decoration by reversing ellipticity of  $\hat{e}_{x_4}$ ,  $\check{e}_{x_6}$  and  $\check{h}_{x_2}$  in each respective region.

We now relate lattice paths and lattice regions to an orbital moment map  $a$ . As mentioned above, the Reeb vector field  $\bar{R}$  of  $\pi_1^* a$  as a function from  $\mathbb{I}$  to  $\mathfrak{t}^2$  is

$$\bar{R} = (a')^\vee / (a \times a')$$

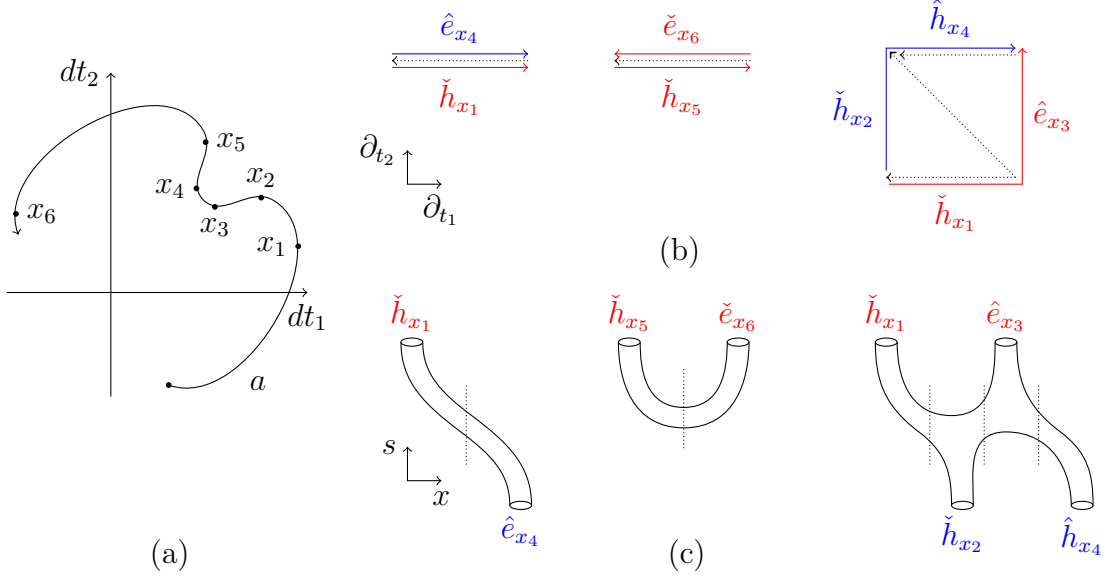


Figure 1: (a) Graph of an orbital moment map  $a$ , (b) regions contributing to the differential, and (c) sketches of corresponding  $J$ -holomorphic curves.

where we identify  $(\mathfrak{t}^2)^* \cong \mathfrak{t}^2$  via  $(dt_1)^\vee = -\partial_{t_2}$  and  $(dt_2)^\vee = \partial_{t_1}$ . It is also convenient to write  $u_1 \sim u_2$  when  $u_1$  is a positive multiple of  $u_2$  for  $u_1, u_2 \in \mathfrak{t}^2, (\mathfrak{t}^2)^*$  or  $\mathbb{R}$  and  $\delta_x : \mathbb{I} \rightarrow \mathbb{R}$  for the function supported at  $\{x\}$  with  $\delta_x(x) = 1$ .

**Definition 1.5.** Let  $a$  be a generic orbital moment map and  $\lambda$  a perturbation of  $\pi_{\mathbb{I}}^* a$  described earlier (and more precisely in §2.2).

(a) A path  $\bar{\mathcal{P}}$  is  $a$ -compatible if  $v_{\bar{\mathcal{P}}}(x) \sim a'(x)^\vee$  and  $c_{\bar{\mathcal{P}}}(x) \sim (a' \times a'')(x)$  for each  $x \in \text{supp } \bar{\mathcal{P}}$ .

(b) A region  $\bar{\mathcal{R}}$  is  $a$ -positive if each  $\bar{\mathcal{P}}^i$  is  $a$ -compatible and  $(a')^\vee \times \sigma_{\bar{\mathcal{R}}} \geq 0$ .

To an orbit set  $\gamma = \{(e_{x_i}, m_i^e), (h_{x_j}, m_j^h)\}$  of  $\lambda$ , we associate  $\mathcal{P}_\gamma$ , the unique  $a$ -compatible decorated path with  $m_{\mathcal{P}_\gamma}^e = \sum m_i^e \delta_{x_i}$  and  $m_{\mathcal{P}_\gamma}^h = \sum m_j^h \delta_{x_j}$ . To a pair of orbit sets  $\alpha$  and  $\beta$  with  $[\alpha] = [\beta] \in H_1(T^2)$ , we associate the decorated region  $\mathcal{R}_{\alpha, \beta} = (\mathcal{P}_\alpha, \mathcal{P}_\beta)$ .

Figure 1 shows an orbital moment map  $a$  and  $a$ -positive lattice regions associated to admissible orbit sets  $\alpha$  and  $\beta$  of  $\lambda$ . It also sketches  $J$ -holomorphic curves  $C$  from  $\alpha$  to  $\beta$  with each dotted line showing the “slice”  $C \cap (\mathbb{R} \times \{x_0\} \times T^2)$ , whose homology class agrees with  $\sigma_{\mathcal{R}}(x_0)$ . According to Theorem 1.1, these regions correspond to non-zero differential coefficients. For non-examples, see Figure 2.

**Remark 1.6.** Write  $(\mathcal{P}^0, \mathcal{P}^1)$  for  $\mathcal{R}'$  from Theorem 1.1. If  $a$  is convex everywhere, e.g.  $(T^3, \lambda_n)$ , it is easy to deduce from Definition 1.4 that  $m(\mathcal{P}^0) = 2$  with  $\text{supp}(\mathcal{P}^0) = \partial(\text{supp } \mathcal{R}')$ . This is the “rounding corner” operation in [11]. Similarly, if  $a$  is concave everywhere,  $m(\mathcal{P}^1) = 2$  with  $\text{supp}(\mathcal{P}^1) = \partial(\text{supp } \mathcal{R}')$ , a “dual” to rounding a corner as in [10]. In general, if  $a_1$  and  $a_2$  are orbital moment maps and  $a_2$  is a reflection of  $a_1$  through a line of rational slope in  $(\mathfrak{t}^2)^*$ , ECC of  $\pi_{\mathbb{I}}^* a_2$  is dual to ECC of  $\pi_{\mathbb{I}}^* a_1$ .

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## 2 Preliminaries

### 2.1 Review of embedded contact homology

We briefly review ECH following [9] (see also [6]). Let  $Y$  be a 3-manifold with a non-degenerate contact form  $\lambda$  and pick a generic  $\lambda$ -admissible almost complex structure  $J$  on  $\mathbb{R} \times Y$ . Admissibility means that  $J$  is  $\mathbb{R}$ -invariant, sends  $\xi = \ker \lambda$  to itself rotating positively with respect to  $d\lambda$ , and  $J(\partial_s) = R$  where  $s$  is the  $\mathbb{R}$ -coordinate and  $R$  is the Reeb vector field.

**Generators.** An *orbit set*  $\gamma$  is a finite set of pairs  $\{(\gamma_i, m_i)\}$  where  $\gamma_i$  are distinct embedded Reeb orbits and  $m_i$  are positive integers. We say  $\gamma$  is *admissible* if  $m_i = 1$  whenever  $\gamma_i$  is hyperbolic and its homology class is  $\sum_i m_i [\alpha_i] \in H_1(Y)$ . The ECH chain complex  $ECC(Y, \lambda, J)$  (or  $ECC(Y, \lambda, J, \Gamma)$ ) is generated (over  $\mathbb{Z}/2$  coefficients) by admissible orbit sets (in the homology class  $\Gamma$ ).

**Holomorphic currents.** Consider  $J$ -holomorphic curves in  $(\mathbb{R} \times Y, J)$  with positive and negative ends at Reeb orbits. Two  $J$ -holomorphic curves  $C$  and  $C'$  are said to be *equivalent* if  $C$  is obtained from  $C'$  by a pre-composition with a biholomorphic map on its domain. Then, a  *$J$ -holomorphic current*  $\mathcal{C}$  is a finite set of pairs  $\{(C_k, d_k)\}$  where  $C_k$  are equivalent classes of distinct irreducible somewhere injective  $J$ -holomorphic curves and  $d_k$  are positive integers. The *moduli space*  $\mathcal{M}^J(\alpha, \beta)$  (or  $\mathcal{M}(\alpha, \beta)$ ) of  $J$ -holomorphic currents from  $\alpha = \{(\alpha_i, m_i)\}$  to  $\beta = \{(\beta_j, n_j)\}$  consists of  $J$ -holomorphic currents whose total multiplicity of positive ends at covers of  $\alpha_i$  is  $m_i$  and whose total multiplicity of negative ends at covers of  $\beta_i$  is  $n_i$ , with no other ends. The homology class of  $\mathcal{C}$  is  $\sum_k d_k [C_k]$ . We say that  $\mathcal{C}$  is *somewhere injective* if  $d_k = 1$  for each  $k$  and that  $\mathcal{C}$  is *embedded* if it is somewhere injective, each  $C_k$  is

embedded and  $C_k$  are pairwise disjoint.

**The ECH index.** For  $\alpha$  and  $\beta$  as above, let  $H_2(Y, \alpha, \beta)$  denote the set of 2-chains  $Z$  in  $Y$  with  $\partial Z = \sum_i m_i \alpha_i - \sum_j n_j \beta_j$ , modulo boundaries of 3-chains. Fix a symplectic trivialization  $\tau$  of  $\xi$  over each  $\alpha_i$  and  $\beta_j$ . The *ECH index* for the triple  $(\alpha, \beta, Z)$  is

$$I(\alpha, \beta, Z) := c_\tau(Z) + Q_\tau(Z) + CZ_\tau^I(\alpha, \beta) \in \mathbb{Z}. \quad (1)$$

Here,

$$CZ_\tau^I(\alpha, \beta) := \sum_i \sum_{k=1}^{m_i} CZ_\tau(\alpha_i^k) - \sum_j \sum_{l=1}^{n_j} CZ_\tau(\beta_j^l)$$

where  $CZ_\tau(\rho) \in \mathbb{Z}$  denotes the Conley-Zehnder index of an orbit  $\rho$  with respect to  $\tau$ . If  $S$  is an embedded surface representative of  $Z$ , the *relative Chern class*  $c_\tau(Z) = \langle c_1(\xi, \tau), Z \rangle$  is the count of zeroes of a section of  $\xi|_S$  which is constant with respect to  $\tau$  near each of its ends. The *relative intersection pairing*  $Q_\tau(Z)$  is the count of intersections of two embedded (except at the boundary) transversely intersecting surfaces  $S$  and  $S'$  in  $[-1, 1] \times Y$  subject to the following: (i)  $S$  and  $S'$  represent  $Z$  and  $\partial S = \partial S' = \sum_i m_i (\{1\} \times \alpha_i) - \sum_j n_j (\{-1\} \times \beta_j)$  and (ii) the projection of  $(S \cup S') \cap ((1 - \epsilon, 1) \times Y)$  to  $Y$  is an embedding, and its image in a transverse slice to  $\alpha_i$  is a union of rays which do not intersect and which do not rotate with respect to  $\tau$  as one goes around  $\alpha_i$  (and similarly for  $(S \cup S') \cap ((-1, -1 + \epsilon) \times Y)$ ).

We remark that  $I(\alpha, \beta, Z)$  does not depend on the choice of  $\tau$ . If  $C$  (or  $\mathcal{C}$ ) is a  $J$ -holomorphic curve (current) from  $\alpha$  to  $\beta$  in the homology class  $Z \in H_2(Y, \alpha, \beta)$ , we also write  $I(C)$  (or  $I(\mathcal{C})$ ) for  $I(\alpha, \beta, Z)$ . Compare (1) with the Fredholm index

$$\text{ind}(C) = -\chi(\Sigma) + 2c_\tau([C]) + \sum CZ_\tau(\rho_i^+) - \sum CZ_\tau(\rho_j^-), \quad (2)$$

where  $\Sigma$  is the domain of  $C$  and the two sums are over its positive ends at  $\rho_i^+$  and negative ends at  $\rho_j^-$ , respectively.

**Proposition 2.1.** ([9, §3]) Let  $\alpha, \beta$  and  $\gamma$  be orbit sets of  $\lambda$  in the homology class  $\Gamma$ .

(a) If  $Z, Z' \in H_2(Y, \alpha, \beta)$  and  $W \in H_2(Y, \beta, \gamma)$ ,

$$I(\alpha, \beta, Z) - I(\alpha, \beta, Z') = \langle c_1(\xi) + 2PD(\Gamma), Z - Z' \rangle \quad (3)$$

where  $PD$  denotes the Poincare dual and

$$I(\alpha, \gamma, Z + W) = I(\alpha, \beta, Z) + I(\beta, \gamma, W).$$

(b) If  $C \in \mathcal{M}(\alpha, \beta)$  is somewhere injective,

$$\text{ind}(C) \leq I(C) \quad (4)$$

with equality only if  $C$  is embedded and satisfies the partition condition below.

(c) If  $\mathcal{C} \in \mathcal{M}(\alpha, \beta)$  contains no trivial cylinders and  $\mathcal{T}$  is a union of trivial cylinders,

$$I(\mathcal{C} \cup \mathcal{T}) \geq I(\mathcal{C}) + 2\#(\mathcal{C} \cap \mathcal{T}). \quad (5)$$

**Partition conditions.** Let  $C$  be a  $J$ -holomorphic curve from  $\alpha = \{(\alpha_i, m_i)\}$  to  $\beta = \{(\beta_j, n_j)\}$ . For each  $i$ ,  $C$  has ends at covers of  $\alpha_i$  with total multiplicity  $m_i$ . The multiplicities of these covers form a partition of  $m_i$ , which we denote by  $p_i^+(C)$ . We similarly define the partition  $p_j^-(C)$  of  $n_j$  for each  $j$ .

For each embedded Reeb orbit  $\rho$  and each positive integer  $m$ , we define two special partitions  $p_\rho^+(m)$  and  $p_\rho^-(m)$  of  $m$ . If  $\rho$  is positive hyperbolic, then  $p_\rho^+(m) = p_\rho^-(m) = (1, \dots, 1)$ . If  $\rho$  is elliptic with rotation angle  $\phi$ , let  $\Lambda_\phi^+(m)$  be the maximal concave polygonal path in the  $x, y$ -plane with vertices at lattice points which starts at the origin, ends at  $(m, \lfloor m\phi \rfloor)$  and lies below the line  $y = \phi x$ . Then,  $p_\rho^+(m) = p_\phi^+(m)$  consists of the horizontal displacements of the segments of  $\Lambda_\phi^+(m)$  connecting consecutive lattice points; and  $p_\rho^-(m) := p_{-\phi}^-(m)$ . (For more details or  $p_\rho^\pm$  for negative hyperbolic  $\rho$ , see [9, §3.9].) Any  $C$  satisfying equality in (4) must satisfy  $p_i^+(C) = p_{\alpha_i}^+(m_i)$  and  $p_j^-(C) = p_{\beta_j}^-(n_j)$  for each  $i$  and  $j$ .

**Differentials.** Let  $\mathcal{M}_k(\alpha, \beta) := \{C \in \mathcal{M}(\alpha, \beta) | I(C) = k\}$ . The key consequence of (4) and (5) is that, if  $J$  is generic, any  $C \in \mathcal{M}_1(\alpha, \beta)$  can be written as the disjoint union  $C' \sqcup \mathcal{T}$  where  $\mathcal{T}$  is trivial and  $C'$  is an irreducible embedded  $J$ -holomorphic curve with  $I(C') = \text{ind}(C') = 1$ . We also have that  $\mathcal{M}_1(\alpha, \beta)/\mathbb{R}$  is compact by a version of Gromov compactness (See [9, §5.3]). If  $\alpha$  and  $\beta$  are admissible, we define

$$\langle \partial\alpha, \beta \rangle := \#(\mathcal{M}_1(\alpha, \beta)/\mathbb{R}) \in \mathbb{Z}/2.$$

**Filtration.** The action  $\mathcal{A}(\alpha)$  of an orbit set  $\alpha = \{(\alpha_i, m_i)\}$  is

$$\mathcal{A}(\alpha) := \sum_i m_i \int_{\alpha_i} \lambda.$$

By Stokes' theorem, the ECH chain complex is filtered by the action of its generators. For each  $L > 0$ , the filtered ECH chain complex  $ECC^L$  is generated only by orbit sets with action less than  $L$ . In this paper, we formulate the filtered ECC so that there is a natural chain inclusion map  $ECC^L(Y, \lambda_L, J_L) \rightarrow ECC^{L'}(Y, \lambda_{L'}, J_{L'})$  for  $L < L'$ . We recover the full ECH as the direct limit of  $ECH^L$  as  $L \rightarrow \infty$ . With this understood, we drop  $L$  from the notation.



## 2.2 Morse-Bott theory

We return to  $Y = \mathbb{I} \times T^2$  with a contact form  $\pi_{\mathbb{I}}^* a$ . To define ECC, we perturb  $\pi_{\mathbb{I}}^* a$  to a non-degenerate  $\lambda$  and choose a generic  $\lambda$ -admissible almost complex structure  $J$  on  $\mathbb{R} \times Y$ . The goal of this section is to describe  $\lambda$  and  $J$  for which ECC yields a nice combinatorial description.

Define  $\bar{Q} : \mathbb{I} \rightarrow \mathfrak{t}^2$ ,  $v_a : \mathbb{I} \rightarrow \Lambda$  and  $\mathcal{A}_a : \mathbb{I} \rightarrow \mathbb{R}^+ \cup \{\infty\}$  by  $\bar{Q} := -a^\vee$ ;  $v_a(x) = v$  if  $a'(x)^\vee \sim v$  for a primitive  $v \in \Lambda$  and 0 otherwise; and  $\mathcal{A}_a(x) = \frac{(a \times a')(x)}{\|a'(x)\|} \|v_a(x)\|$  if  $v_a(x) \neq 0$  and infinity otherwise. Let  $\Xi_L := \{x \in \mathbb{I} \mid \mathcal{A}_a(x) < L\}$  and  $N := L / \min \mathcal{A}_a$ . Then, for  $\rho$  in some  $S^1$ -family  $\bar{\rho}_x$ ,  $\mathcal{A}(\rho) = \mathcal{A}_a(x)$ ,  $\mathcal{L}_{\partial_x} \bar{R} = \frac{a' \times a''}{(a \times a')^2} \bar{Q}$  and  $\mathcal{L}_{\bar{Q}} \bar{R} = 0$ , so the linearized Reeb flow over  $\rho$  is contained in the Maslov cycle with the return map

$$\begin{pmatrix} 1 & 0 \\ \frac{a' \times a''}{(a \times a')^2} \mathcal{A}_a & 1 \end{pmatrix}, \quad (6)$$

while  $\bar{Q}$  describes a section of  $\xi$  with  $(d\lambda)(\partial_x, \bar{Q}) = a \times a' > 0$ , giving a trivialization  $\tau$  of  $\xi$  by

$$\xi \cong \text{span}\{\partial_x, \bar{Q}\}.$$

**Perturbation.** (cf. [11, §10.5]) Whenever  $v_a(x) = (p, q) \neq 0$ , define  $\Theta_x : \{x\} \times T^2 \rightarrow \mathbb{R}/\mathbb{Z}$  by

$$\Theta_x(x, t_1, t_2) = (t_1, t_2) \times (p, q) + pq/2.$$

For each  $\theta \in \mathbb{R}/\mathbb{Z}$ ,  $\Theta_x^{-1}(\theta)$  is the image of a unique embedded orbit in  $\bar{\rho}_x$ , which we denote by  $\bar{\rho}_x(\theta)$ . Fix  $\eta > 0$ ,  $0 < \theta_h < 1/5$  and  $\theta_e := -\theta_h/N$ . For each  $x \in \Xi_L$ , let  $f_x : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$  be a Morse function which attains maximum at  $\theta_x^{\max}$  and minimum at  $\theta_x^{\min}$  with no other critical points, where  $(\theta_x^{\max}, \theta_x^{\min}) = (\theta_e, \theta_h)$  if  $\bar{\rho}_x$  is convex and  $(-\theta_h, -\theta_e)$  otherwise. Then, choose disjoint neighbourhoods  $U_x$  of  $x$  so that  $a' \times a''$  does not vanish on  $U_x$  and extend  $\Theta_x^*(f_x)$  to a function  $\tilde{f}_x$  on  $U_x \times T^2$  with a compact support and  $\partial \tilde{f}_x / \partial x = 0$  near  $\{x\} \times T^2$ . If  $\eta$  is sufficiently small,

$$\lambda := (1 + \eta \tilde{f}_x) \pi_{\mathbb{I}}^* a \quad (7)$$

is a contact form on  $U_x \times T^2$  with non-degenerate embedded orbits  $\bar{\rho}_x(\theta_e)$  and  $\bar{\rho}_x(\theta_h)$  and no other embedded orbits of action less than  $L$  [1]. By (6), if  $\bar{\rho}_x$  is convex,  $\check{e}_x := \bar{\rho}_x(\theta_e)$  is elliptic,  $\check{h}_x := \bar{\rho}_x(\theta_h)$  is hyperbolic and their  $m$ -fold iterates for  $m < N$  have  $CZ_\tau(\check{e}_x^m) = 1$  and  $CZ_\tau(\check{h}_x^m) = 0$ , provided  $\eta$  is sufficiently small. Similarly, if  $\bar{\rho}_x$  is concave,  $\hat{e}_x := \bar{\rho}_x(\theta_e)$  is elliptic with  $CZ_\tau(\hat{e}_x^m) = -1$  and  $\hat{h}_x := \bar{\rho}_x(\theta_h)$  is hyperbolic with  $CZ_\tau(\hat{h}_x^m) = 0$ .

**Definition 2.2.** We say a perturbation  $\lambda$  of  $\pi_{\mathbb{I}}^* a$  is *good* if it is of the form (7) on  $U_x \times T^2$  for each  $x \in \Xi_L$  and unperturbed elsewhere.

**Holomorphic building.** Define an almost complex structure  $\bar{J}$  on  $\mathbb{R} \times \mathbb{I} \times T^2$  by  $\bar{J}(\partial_s) = \bar{R}$  and  $\bar{J}(\partial_x) = \bar{Q}$ . For  $(Y, \pi_{\mathbb{I}}^* a, \bar{J}, \{f_x\}_{x \in \Xi_L})$ , a  $\bar{J}$ -holomorphic building  $\bar{C}$  is a sequence of  $\bar{J}$ -holomorphic curves  $(\bar{C}^1, \dots, \bar{C}^l)$  such that:

- (i) Each end of  $\bar{C}^i$  converges to the  $m$ -fold iterate  $\bar{\rho}_x^m(\theta)$  of some  $\bar{\rho}_x(\theta)$ .
- (ii) For  $1 \leq i < l$ , there is a bijective pairing between the negative ends of  $\bar{C}^i$  and the positive ends of  $\bar{C}^{i+1}$ . For each such pair, the negative end of  $\bar{C}^i$  converges to  $\bar{\rho}_x^m(\theta^-)$ , the positive end of  $\bar{C}^{i+1}$  converges to  $\bar{\rho}_x^m(\theta^+)$  for the same  $\bar{\rho}_x$  and  $m$  and there is a downward flow of  $f_x$  from  $\theta^-$  to  $\theta^+$ .
- (iii) For each positive end of  $\bar{C}^1$  at some  $\bar{\rho}_x^m(\theta^+)$ , there is a downward flow of  $f_x$  from a critical point of  $f_x$  to  $\theta^+$ . For each negative end of  $\bar{C}^l$  at some  $\bar{\rho}_x^m(\theta^-)$ , there is a downward flow of  $f_x$  from  $\theta^-$  to a critical point of  $f_x$ .

### 3 Proof of the main theorem

**Definition 3.1.** The *local combinatorial ECH index* of a decorated region  $\mathcal{R}$  at  $x$  is  $I_{\mathcal{R}}(x) := Q_{\mathcal{R}}(x) + CZ_{\mathcal{R}}(x)$  where

$$Q_{\mathcal{R}}(x) = m_{\mathcal{R}}(x) \cdot (v_{\mathcal{R}}(x) \times \sigma_{\mathcal{R}}(x))$$

and

$$CZ_{\mathcal{R}}(x) = c_{\mathcal{P}^0}(x) \cdot m_{\mathcal{P}^0}^e(x) - c_{\mathcal{P}^1}(x) \cdot m_{\mathcal{P}^1}^e(x).$$

The *combinatorial ECH index*  $I(\mathcal{R})$  of  $\mathcal{R}$  is  $\sum_x I_{\mathcal{R}}(x)$ .

Note  $\sum_x Q_{\mathcal{R}}(x)$  is the area of the polygon depicting  $\mathcal{R}$  with respect to the standard area form. As one might expect:

**Proposition 3.2.** Let  $a$  be a generic orbital moment map and  $\lambda$  a good perturbation of  $\pi_{\mathbb{I}}^* a$ . For orbit sets  $\alpha$  and  $\beta$  of  $\lambda$  with  $[\alpha] = [\beta]$  and any  $Z \in H_2(Y, \alpha, \beta)$ ,

$$I(\mathcal{R}_{\alpha, \beta}) = I(\alpha, \beta, Z).$$

*Proof.* Since  $\xi$  is trivial and the generator  $[T^2] \in H_2(Y)$  has algebraic intersection number zero with every orbit,  $c_{\tau}(Z) = 0$  and  $I(\alpha, \beta, Z)$  is independent of  $Z$  by (3). It is also clear that  $\sum_x CZ_{\mathcal{R}_{\alpha, \beta}}(x) = CZ_{\tau}^I(\alpha, \beta)$  (see §2.2). To compute  $Q_{\tau}(Z)$ , let

$$G := (\{0\} \times \mathbb{I}) \cup \{(s, x + (1 - |s|)\epsilon)\}_{s \in [-1, 1], x \in \text{supp}(\mathcal{P}_{\alpha} \cup \mathcal{P}_{\beta})} \quad (8)$$

be a union of line segments in  $[-1, 1] \times \mathbb{I}$  with multivalent vertices

$$V := \{(0, x + \epsilon)\}_{x \in \text{supp}(\mathcal{P}_{\alpha} \cup \mathcal{P}_{\beta})}.$$

Let  $B_{\epsilon/2}(V)$  be the  $(\epsilon/2)$ -neighbourhood of  $V$  and  $\pi := \text{id}_{[-1,1]} \times \pi_{\mathbb{I}}$ . We want a smooth surface  $S \subset [-1,1] \times Y$  as in §2.1 such that (i)  $\pi(S) \subset G \cup B_{\epsilon/2}(V)$  and (ii) for each component  $E$  of  $G \setminus B_{\epsilon/2}(V)$ ,  $\pi|_S^{-1}(E)$  consists of minimal number of embedded (disjoint except at  $\{\pm 1\} \times Y$ )  $v$ -invariant cylinders, where  $v = \sigma_{\mathcal{R}_{\alpha,\beta}}(x)$  if  $(0, x) \in E$  and  $v_a(x)$  if  $(\pm 1, x) \in E$ . We can construct such an  $S$  by gluing these cylinders so that, near each  $(0, x_0) \in V$ , the projection  $\mathcal{Z}_x$  of  $S \cap ([-1,1] \times \{x\} \times T^2)$  to  $T^2$  is a movie of curves with  $\mathcal{Z}_{x_0+\epsilon/2}$  obtained from  $\mathcal{Z}_{x_0-\epsilon/2}$  by resolving intersections.

If  $\psi$  is an automorphism of  $[-1,1]$  with  $\psi(-1+\epsilon) = 0$  and  $\psi(0) = 1-\epsilon$ ,  $S$  and  $S' := (\psi \times \text{id}_Y)(S)$  intersect in  $\pi^{-1}(1-\epsilon, x+\epsilon^2)$  with signed count  $m_{\mathcal{P}_\alpha}(x) \cdot (v_{\mathcal{P}_\alpha}(x) \times \sigma_{\mathcal{R}_{\alpha,\beta}}(x))$  for each  $x \in \text{supp } \mathcal{P}_\alpha$ , and in  $\pi^{-1}(0, x+\epsilon^2)$  with signed count  $m_{\mathcal{P}_\beta}(x) \cdot (v_{\mathcal{P}_\beta}(x) \times \sigma_{\mathcal{R}_{\alpha,\beta}}(x))$  for each  $x \in \text{supp } \mathcal{P}_\beta$ . We get  $\sum_x Q_{\mathcal{R}_{\alpha,\beta}}(x) = Q_\tau(Z)$  by summing up these numbers.  $\square$

### 3.1 Positivity

**Lemma 3.3.** Let  $a$  be a generic orbital moment map and  $\mathcal{R}$  a lattice region.

- (a) If  $\mathcal{R}$  is positive, indecomposable and non-local,  $I_{\mathcal{R}} \geq 0$ .
- (b) If  $\bar{\mathcal{R}}$  is  $a$ -positive, then it is positive and it decomposes at  $x$  whenever  $a'(x)^\vee \times \sigma_{\bar{\mathcal{R}}}(x) = 0$ .

*Proof.* Part (a) is clear from definition. For (b), we write  $\bar{\mathcal{R}} = (\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^1)$  and show that if  $\bar{\mathcal{R}}$  is non-local and indecomposable, then: (i)  $Q_{\bar{\mathcal{R}}} \geq 0$ , (ii)  $(a')^\vee \times \sigma_{\bar{\mathcal{R}}} > 0$  on  $\text{int}(\text{supp } \bar{\mathcal{R}})$  and (iii)  $c_{\bar{\mathcal{P}}^0}(x) \geq 0$  and  $c_{\bar{\mathcal{P}}^1}(x) \leq 0$  for  $x \in \partial(\text{supp } \bar{\mathcal{R}})$ . Claim (i) follows from  $a$ -positivity at  $x \in \text{supp}(m_{\bar{\mathcal{R}}})$ . For (ii) and (iii), suppose  $a'(x_0)^\vee \times \sigma_{\bar{\mathcal{R}}}(x_0) = 0$  for  $x_0 \in \text{supp}(\bar{\mathcal{R}})$  and write  $\sigma_{\bar{\mathcal{R}}}(x_0 \pm \epsilon) = b_\pm \cdot a'(x_0)^\vee$ . By  $a$ -positivity,  $a'(x_0 \pm \epsilon) \times b_\pm a'(x_0) \geq 0$ , while  $a'(x_0) \times a'(x_0 \pm \epsilon) \sim \pm(a' \times a'')(x_0)$ . In particular,  $b_+ b_- \leq 0$  so  $x_0 \in \partial(\text{supp } \bar{\mathcal{R}})$  by indecomposability. Furthermore, if  $a$  is convex at  $x_0$ ,  $b_+ - b_- < 0$  so  $m_{\bar{\mathcal{P}}^0}(x_0) > 0$  and by non-locality,  $m_{\bar{\mathcal{P}}^1}(x_0) = c_{\bar{\mathcal{P}}^1}(x_0) = 0$ . Similarly, if  $a$  is concave at  $x_0$ ,  $c_{\bar{\mathcal{P}}^0}(x_0) = 0$ .  $\square$

**Definition 3.4.** Let  $a$  be a generic orbital moment map and let  $\lambda$  be  $\pi_{\mathbb{I}}^* a$  or a good perturbation thereof. To each orbit set  $\gamma$  of  $\lambda$ , we associate  $\bar{\mathcal{P}}_\gamma$ , the unique  $a$ -compatible path such that  $\gamma$  contains  $m_{\bar{\mathcal{P}}_\gamma}(x)$  orbits (counted with multiplicity) at  $x$ . To orbit sets  $\alpha$  and  $\beta$  with  $[\alpha] = [\beta]$ , we associate  $\bar{\mathcal{R}}_{\alpha,\beta} := (\bar{\mathcal{P}}_\alpha, \bar{\mathcal{P}}_\beta)$ .

**Lemma 3.5.** Let  $\lambda$  be as in Definition 3.4 and  $J$  a generic  $\lambda$ -admissible almost complex structure. If  $\alpha$  and  $\beta$  are orbit sets of  $\lambda$  and  $\mathcal{M}(\alpha, \beta)$  is nonempty,  $\bar{\mathcal{R}}_{\alpha,\beta}$  is  $a$ -positive. Moreover, if  $C \in \mathcal{M}(\alpha, \beta)$  has no end at  $x \in \mathbb{I}$  and  $\lambda$  is unperturbed near  $\{x\} \times T^2$ , then  $a'(x)^\vee \times \sigma_{\bar{\mathcal{R}}_{\alpha,\beta}}(x) = 0$  if and only if  $\mathcal{S}_x := C \cap (\mathbb{R} \times \{x\} \times T^2) = \emptyset$ .

*Proof.* If  $\lambda$  is unperturbed near  $\{x_0\} \times T^2$  and  $C$  has no end at  $x_0$ , differentiate

$$\langle a(x) - a(x_0), [\mathcal{S}_{x_0}] \rangle = \int_{C \cap (\mathbb{R} \times [x, x_0] \times T^2)} d\lambda \quad (9)$$

near  $x_0$  to get  $a'(x)^\vee \times \sigma_{\bar{\mathcal{R}}}(x) = \langle a'(x), \sigma_{\bar{\mathcal{R}}}(x) \rangle \geq 0$  with equality only if  $\mathcal{S}_{x_0} = \emptyset$  by genericity of  $a$ . The inequality extends to all  $x$  by continuity and since  $a' \times a''$  does not vanish on  $U_x$ .  $\square$

### 3.2 Indecomposability

Let  $\lambda$  be a good perturbation of  $\pi_{\mathbb{I}}^* a$  for a generic orbital moment map  $a$  and  $J$  a generic  $\lambda$ -admissible almost complex structure.

**Proposition 3.6.** Let  $\alpha$  and  $\beta$  be orbit sets of  $\lambda$  and  $C \in \mathcal{M}_1(\alpha, \beta)$ . Then its non-trivial component  $C' \in \mathcal{M}_1(\alpha', \beta')$  has genus 0 and  $\mathcal{R}_{\alpha, \beta} = \mathcal{T}_1 \mathcal{R}_{\alpha', \beta'} \mathcal{T}_2$  where  $\mathcal{R}_{\alpha', \beta'}$  is indecomposable and  $\mathcal{T}_i$  are trivial. Moreover, we have a bijection

$$\mathcal{M}_1(\alpha, \beta) \cong \mathcal{M}_1(\alpha', \beta').$$

*Proof.* By the partition condition, we can rewrite (2) for  $C'$  as

$$\begin{aligned} 1 = \text{ind}(C') &= 2g - 2 + \sum (1 + CZ_\tau(\rho_i^+)) + \sum (1 - CZ_\tau(\rho_j^-)) \\ &= 2g - 2 + \sum (m_{\mathcal{R}'}(x) + CZ_{\mathcal{R}'}(x)). \end{aligned} \quad (10)$$

where  $g$  is the genus of  $C'$  and  $\mathcal{R}' = \mathcal{R}_{\alpha', \beta'}$ . Write  $\text{supp } \mathcal{R}' = [x_1, x_2]$ . If  $x_1 = x_2$ ,  $CZ_{\mathcal{R}'}(x_1) = I(\mathcal{R}') = 1$  so  $2g + m(\mathcal{R}') \leq 2$ , forcing  $g = 0$  and  $m(\mathcal{R}') = 2$ . Otherwise, for each factor  $\mathcal{R}'_i$  of  $\mathcal{R}'$  and  $x \in \partial(\text{supp } \mathcal{R}'_i)$ ,  $m_{\mathcal{R}'_i}(x) + CZ_{\mathcal{R}'_i}(x) \geq 1$ , so  $g = 0$  and  $\mathcal{R}'$  contains one non-local factor and possibly one local factor. We draw a contradiction when it contains a local factor  $\mathcal{R}'_{i_0}$ . By symmetry, we only argue for the case  $\text{supp } \mathcal{R}'_{i_0} = \{x_1\}$  and  $c_{\mathcal{R}'}(x_1) = 1$ , and by  $SL_2(\mathbb{Z})$ -symmetry, assume  $v_{\mathcal{R}'}(x_1) = (0, 1)$ . Let  $S := C' \cap ([-s, s] \times [0, x_1 + \epsilon] \times T^2)$  for generic  $s, 1/\epsilon \gg 0$ . Since  $I(\mathcal{R}') = I(\mathcal{R}_{\check{e}_{x_1}, \check{e}_{x_1}} \mathcal{R}')$ ,  $S$  does not intersect  $\mathbb{R} \times \check{e}_{x_1}$  by (5) and it maps to

$$Y' := ([0, x_1 + \epsilon] \times S^1) / (\{0\} \times S^1) \setminus \{(x_1, \theta_e)\}$$

by  $(s, x, t_1, t_2) \mapsto (x, t_1)$ . Let  $\mathcal{S}_1, \dots, \mathcal{S}_n$  denote the boundary components of  $S$  with  $\mathcal{S}_1 \subset \{-s\} \times Y$  corresponding to the unique negative end of  $C'$  at  $\check{e}_{x_1}$ . Since (the two)  $\mathcal{S}_i \subset \{s\} \times Y$  maps to a neighbourhood of  $(x_1, \theta_h)$  and  $\sigma_{\mathcal{R}'}(x_1 + \epsilon) = (0, -1)$ , the total degree of  $\cup_{i=2}^n \mathcal{S}_i$  mapping to  $Y' \simeq S^1$  is zero. Hence,  $\mathcal{S}_1$  has winding number zero around  $(x_1, \theta_e)$ , contradicting its lower bound of  $\lceil CZ_\tau(\check{e}_{x_1})/2 \rceil$  from the decay condition [5].

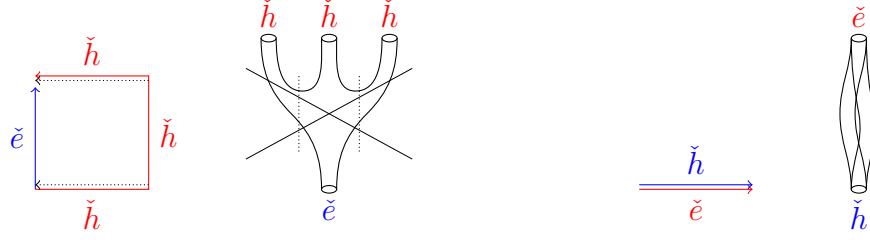


Figure 2: Examples of case (c2) and (c3)

If  $C$  contains a trivial cylinder at  $x_0$ ,  $I(\mathcal{R}_{\alpha,\beta}) = I(\mathcal{R}')$  implies  $Q_{\mathcal{R}_{\alpha,\beta}}(x_0) = Q_{\mathcal{R}'}(x_0)$ . By Lemma 3.3(b),  $x_0 \notin \text{int}(\text{supp } \mathcal{R}')$  and  $\mathcal{R}_{\alpha,\beta} = \mathcal{T}_1 \mathcal{R}' \mathcal{T}_2$  for some  $\mathcal{T}_i$ . To see the bijection, each  $\mathcal{C} \in \mathcal{M}_1(\alpha', \beta')$  gives a distinct member of  $\mathcal{M}_1(\alpha, \beta)$  by unioning with a trivial current. This mapping is onto since the nontrivial component of any  $C \in \mathcal{M}_1(\alpha, \beta)$  cannot have both a positive and a negative end at  $x \in \text{supp}(\mathcal{T}_i)$  by the above.  $\square$

### 3.3 Classification of $\mathcal{R}_{\alpha,\beta}$ with nonempty $\mathcal{M}_1(\alpha, \beta)$

**Definition 3.7.** (a) The *(local) Morse-Bott ECH index* of a region  $\bar{\mathcal{R}}$  is  $I(\mathcal{R}_{\min})$  (or  $I_{\mathcal{R}_{\min}}$ ) where  $\mathcal{R}_{\min}$  is the minimal decoration of  $\bar{\mathcal{R}}$ . The *(local) Morse-Bott ECH index* of a decorated region is that of its underlying undecorated region.

(b) The *loose multiplicity*  $m_{\bar{\mathcal{R}}}^l(x)$  of  $\bar{\mathcal{R}} = (\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^1)$  at  $x$  is  $m_{\bar{\mathcal{P}}^0}(x)$  if  $c_{\bar{\mathcal{R}}}(x) = 1$  and  $m_{\bar{\mathcal{P}}^1}(x)$  otherwise. We also write  $m^l(\bar{\mathcal{R}}) = \sum_x m_{\bar{\mathcal{R}}}^l(x)$ .

Let  $\alpha$  and  $\beta$  be orbit sets of  $\lambda$  as in §3.2. To study  $\mathcal{M}_1(\alpha, \beta) \neq \emptyset$ , it suffices to assume  $\mathcal{R} = \mathcal{R}_{\alpha,\beta}$  is  $a$ -positive and indecomposable by Lemma 3.5 and Proposition 3.6. Let  $\mathcal{R}_{\min}$  be the minimal decoration of the underlying undecorated region of  $\mathcal{R}$  so  $I(\mathcal{R}) \geq \bar{I}(\mathcal{R}) = I(\mathcal{R}_{\min}) \geq -1$  by Lemma 3.3(a). Hence, we classify  $\mathcal{R}$  by:

- (c1)  $\bar{I}(\mathcal{R}) = 0$ : By positivity of  $I_{\mathcal{R}_{\min}}$ ,  $v_{\mathcal{R}} \times \sigma_{\mathcal{R}} \leq 1$ . If some  $\sigma_{\mathcal{R}}(x)$  is non-primitive,  $Q_{\mathcal{R}} \equiv 0$  so  $\sum(CZ_{\mathcal{R}}(x) + m_{\mathcal{R}}(x)) \geq 4$ , violating (10). Thus,  $\mathcal{R}$  is minimally positive and almost minimally decorated. We claim  $\#(\mathcal{M}_1(\alpha, \beta)/\mathbb{R}) = 1$ .
- (c2)  $\bar{I}(\mathcal{R}) = 1$ : By positivity of  $I_{\mathcal{R}_{\min}}$ , there is a unique  $x_0 \in \text{int}(\text{supp } \mathcal{R})$  such that either (i)  $m_{\mathcal{R}}^l(x_0) = 1$  or (ii)  $v_{\mathcal{R}}(x_0) \times \sigma_{\mathcal{R}}(x_0) = 2$ . In case (ii), one of  $\sigma_{\mathcal{R}}(x_0 \pm \epsilon)$  is twice a primitive vector so  $m_{\mathcal{R}}^l(x_1) = 2$  for some  $x_1 \in \partial(\text{supp } \mathcal{R})$ . Either way,  $m^l(\mathcal{R}) = 3$  and we claim  $\mathcal{M}_1(\alpha, \beta)$  is empty if  $(\lambda, J)$  is close to  $(\pi_{\mathbb{I}}^* a, \bar{J})$ .
- (c3)  $\bar{I}(\mathcal{R}) = -1$ :  $\mathcal{R}$  is a local bigon at  $x_0$ . We claim  $J$ -holomorphic curves in  $\mathcal{M}_1(\alpha, \beta)$  correspond to index 1 Morse flows of  $f_{x_0}$  and, thus, exist in pairs.

We deal with (c2) and (c3) (see Figure 2) in §3.4 and (c1) in the rest of §3.

### 3.4 A Morse-Bott argument

**Definition 3.8.** Let  $\bar{\mathcal{R}} = (\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^2)$ ,  $\bar{\mathcal{R}}^1 = (\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^1)$  and  $\bar{\mathcal{R}}^2 = (\bar{\mathcal{P}}^1, \bar{\mathcal{P}}^2)$  be three regions. The *sharing multiplicity* between  $\bar{\mathcal{R}}^1$  and  $\bar{\mathcal{R}}^2$  at  $x$  is

$$m_{\bar{\mathcal{R}}^1, \bar{\mathcal{R}}^2}^s(x) := m_{\bar{\mathcal{P}}^1}(x) + m_{\bar{\mathcal{R}}}^{triv}(x) - m_{\bar{\mathcal{R}}^1}^{triv}(x) - m_{\bar{\mathcal{R}}^2}^{triv}(x)$$

where  $m_{\bar{\mathcal{R}}}^{triv}(x)$  denotes the number of local bigon factors of  $\bar{\mathcal{R}}$  at  $x$ . We also write  $m^{triv}(\bar{\mathcal{R}}) = \sum_x m_{\bar{\mathcal{R}}}^{triv}(x)$  and  $m^s(\bar{\mathcal{R}}^1, \bar{\mathcal{R}}^2) = \sum_x m_{\bar{\mathcal{R}}^1, \bar{\mathcal{R}}^2}^s(x)$ .

When  $\bar{\mathcal{P}}^0$ ,  $\bar{\mathcal{P}}^1$  and  $\bar{\mathcal{P}}^2$  are drawn with each starting at  $0 \in \mathfrak{t}^2$ ,  $m_{\bar{\mathcal{R}}^1, \bar{\mathcal{R}}^2}^s(x)$  is the number of edges of  $\mathcal{P}^1$  at  $x$  “sandwiched” between non-local factors of  $\bar{\mathcal{R}}^1$  and  $\bar{\mathcal{R}}^2$ .

**Lemma 3.9.** For  $\bar{\mathcal{R}}$ ,  $\bar{\mathcal{R}}^1$  and  $\bar{\mathcal{R}}^2$  as above,  $m_{\bar{\mathcal{R}}^1, \bar{\mathcal{R}}^2}^s \geq 0$ . If  $\bar{\mathcal{R}}$  is indecomposable and non-local,

$$\bar{I}(\bar{\mathcal{R}}) - \sum \bar{I}(\bar{\mathcal{R}}_k^i) = m^s(\bar{\mathcal{R}}^1, \bar{\mathcal{R}}^2) = \sum m^l(\bar{\mathcal{R}}_k^i) - m^l(\bar{\mathcal{R}})$$

where each sum is over all non-local factors  $\bar{\mathcal{R}}_k^i$  of  $\bar{\mathcal{R}}^i$  for  $i = 1, 2$ . In particular, if  $\bar{I}(\bar{\mathcal{R}}) = 0$  and  $m^l(\bar{\mathcal{R}}) = 2$ , then either  $\bar{\mathcal{R}}^1$  or  $\bar{\mathcal{R}}^2$  is local.

*Proof.* The first assertion is obvious. For the second, use

$$m_{\bar{\mathcal{R}}}^l - m_{\bar{\mathcal{R}}}^{triv} = \sum m_{\bar{\mathcal{R}}_k^i}^l - m_{\bar{\mathcal{R}}^1, \bar{\mathcal{R}}^2}^s$$

and

$$\bar{I}(\bar{\mathcal{R}}) + m^{triv}(\bar{\mathcal{R}}) = \sum \bar{I}(\bar{\mathcal{R}}_k^i) + m^s(\bar{\mathcal{R}}^1, \bar{\mathcal{R}}^2)$$

which follow from  $m_{\bar{\mathcal{R}}}^l = m_{\bar{\mathcal{R}}^1}^l + m_{\bar{\mathcal{R}}^2}^l - m_{\bar{\mathcal{P}}^1}$  and  $\bar{I}(\bar{\mathcal{R}}) = \bar{I}(\bar{\mathcal{R}}^1) + \bar{I}(\bar{\mathcal{R}}^2) + m(\bar{\mathcal{P}}^1)$ .  $\square$

This is the only section where the choice of  $f_x$  and  $\bar{J}$  in §2.2 plays a role, due to:

**Lemma 3.10.** [11, Proposition 10.16] If  $\bar{C}$  is a  $\bar{J}$ -holomorphic curve with positive ends at  $m_i^+$ -fold covers of  $\bar{\rho}_{x_i^+}(\theta_i^+)$  and negative ends at  $m_j^-$ -fold covers of  $\bar{\rho}_{x_j^-}(\theta_j^-)$ ,

$$\Theta(\bar{C}) := \sum m_i^+ \theta_i^+ - \sum m_j^- \theta_j^- = 0 \in \mathbb{R}/\mathbb{Z}.$$

*Proof.* If a surface  $Z \subset T^2$  bounds  $\bar{\rho}_x(\theta)$  and a representative of  $[\bar{\rho}_x(\theta)]$  contained in  $(S^1 \times \{0\}) \cup (\{0\} \times S^1)$ , then  $\int_Z dt_1 dt_2 = \theta \in \mathbb{R}/\mathbb{Z}$ . Moreover, when restricted to  $(v, \bar{J}v)$  for  $v \in T(\mathbb{R} \times Y)$ ,  $dsdx$  agrees with  $d\bar{R}d\bar{Q} = a \wedge a' / (a \times a') = dt_1 dt_2$ . Hence,

$$\Theta(\bar{C}) = \int_{\bar{C}} dt_1 dt_2 = \lim_{n \rightarrow \infty} \int_{\bar{C}_n} dt_1 dt_2 = \lim_{n \rightarrow \infty} \int_{\bar{C}_n} dsdx = \lim_{n \rightarrow \infty} \int_{\partial \bar{C}_n} (-x ds) = 0$$

where  $\bar{C}_n$  denotes  $\bar{C} \cap ([-n, n] \times Y)$ .  $\square$

**Proposition 3.11.** Let  $\lambda$  be a good perturbation of  $\pi_{\mathbb{I}}^*a$  and  $J$  a generic  $\lambda$ -admissible almost complex structure. Let  $\alpha$  and  $\beta$  be orbit sets of  $\lambda$  and suppose  $\mathcal{R}_{\alpha,\beta}$  is indecomposable with  $I(\mathcal{R}_{\alpha,\beta}) = 1$  and  $\bar{I}(\mathcal{R}_{\alpha,\beta}) = \pm 1$ . Then,  $\#(\mathcal{M}_1(\alpha, \beta)/\mathbb{R}) = 0$  for  $(\lambda, J)$  sufficiently close to  $(\pi_{\mathbb{I}}^*a, \bar{J})$ .

*Proof.* Consider a sequence  $(\lambda_n, J_n)$  of good perturbations of  $\pi_{\mathbb{I}}^*a$  and generic  $\lambda_n$ -admissible  $J_n$  converging to  $(\pi_{\mathbb{I}}^*a, \bar{J})$ . Suppose there exists a  $J_n$ -holomorphic curve  $C_n \in \mathcal{M}_1^{J_n}(\alpha, \beta)$  for each  $n$ . Each  $C_n$  satisfies the partition condition and, by Proposition 3.6, has genus zero. By [1], a subsequence of  $C_n$  converges to a  $\bar{J}$ -holomorphic building  $\bar{C} = (\bar{C}^1, \dots, \bar{C}^l)$  where  $\bar{C}^i \in \mathcal{M}^{\bar{J}}(\alpha^i, \beta^i)$  for orbit sets  $\alpha^i$  and  $\beta^i$  of  $\pi_{\mathbb{I}}^*a$  (see §2.2). Let  $\bar{\mathcal{P}}^{i-1} := \bar{\mathcal{P}}_{\alpha^i}$  and  $\bar{\mathcal{P}}^i := \bar{\mathcal{P}}_{\beta^i}$  for  $1 \leq i \leq l$ . By the definition of  $\bar{J}$ -holomorphic buildings, the two definitions of  $\bar{\mathcal{P}}^i$  agree for  $1 \leq i < l$  and  $\mathcal{R}_{\alpha,\beta}$  decorates  $(\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^l)$ . Write  $\bar{C}^i = \bar{C}_0^{i0} \cup \bar{T}^i$  where  $\bar{T}^i$  is the union of its (multiply covered) trivial cylinder components. By Lemma 3.5, each  $\bar{\mathcal{R}}^i = (\bar{\mathcal{P}}^{i-1}, \bar{\mathcal{P}}^i)$  is  $a$ -positive and  $\bar{C}^i = \bar{T}^i$  whenever  $\bar{\mathcal{R}}^i$  is local.

If  $\bar{I}(\mathcal{R}_{\alpha,\beta}) = -1$ , i.e.  $\mathcal{R}_{\alpha,\beta}$  is a local bigon at some  $x_0$ , then  $\bar{C}$  is a trivial cylinder attached to Morse flows of  $f_{x_0}$ . By [1], for sufficiently large  $n$ ,  $C_n$  is one of the two cylinders corresponding to the two flows from  $\theta_{x_0}^{\max}$  to  $\theta_{x_0}^{\min}$  and  $\#(\mathcal{M}_1^{J_n}(\alpha, \beta)/\mathbb{R}) = 0$ . Henceforth assume  $\bar{I}(\mathcal{R}_{\alpha,\beta}) = 1$ .

*Step 1.* Let  $i_1$  and  $i_2$  be the smallest and the largest  $i$  such that  $\bar{\mathcal{R}}^i$  is non-local. If  $i_1 = i_2$ , each positive (respectively negative) end of  $\bar{C}^{i_1}$  converges to covers of some  $\bar{\rho}_x(\theta_x^{\min})$  (respectively  $\bar{\rho}_x(\theta_x^{\max})$ ). Since  $m^l(\mathcal{R}_{\alpha,\beta}) = 3$  (see §3.3 (c2)),

$$\Theta(\bar{C}^{i_1}) = 3\theta_h + (m(\mathcal{R}_{\alpha,\beta}) - 3)\theta_h/N \neq 0,$$

contradicting Lemma 3.10. Thus,  $i_1 < i_2$ . By Lemma 3.9 on  $(\bar{\mathcal{R}}')^1 := (\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^{i_1})$  and  $(\bar{\mathcal{R}}')^2 := (\bar{\mathcal{P}}^{i_1}, \bar{\mathcal{P}}^l)$ ,  $\sum m^l((\bar{\mathcal{R}}')_k^j) + \sum \bar{I}((\bar{\mathcal{R}}')_k^j) = 4$ . Since each  $(\bar{\mathcal{R}}')^j$  is non-local and  $a$ -positive, each contains exactly one non-local factor with  $m^l((\bar{\mathcal{R}}')_k^j) = 2$  and  $\bar{I}((\bar{\mathcal{R}}')_k^j) = 0$ . By Lemma 3.9 on  $(\bar{\mathcal{P}}^{i_1}, \bar{\mathcal{P}}^{i_2-1})$  and  $(\bar{\mathcal{P}}^{i_2-1}, \bar{\mathcal{P}}^l)$ , we conclude  $\bar{\mathcal{R}}^{i_1} = (\bar{\mathcal{R}}')^1$ ,  $\bar{\mathcal{R}}^{i_2} = (\bar{\mathcal{R}}')^2$ , all other  $\bar{\mathcal{R}}^i$  are local and  $m_{\bar{\mathcal{R}}^{i_1}, \bar{\mathcal{R}}^{i_2}}^s = \delta_{x_0}$  for some  $x_0$ .

*Step 2.* For  $i_0 = i_1$  or  $i_2$ , we claim  $\bar{\mathcal{R}}_0^{i_0} = \bar{\mathcal{R}}_{\alpha_0, \beta_0}$  where  $\bar{\mathcal{R}}_0^{i_0}$  denotes the non-local factor of  $\bar{\mathcal{R}}^{i_0}$  and  $\bar{C}_0^{i_0} \in \mathcal{M}(\alpha_0, \beta_0)$ . Write  $\text{supp } \bar{\mathcal{R}}_0^{i_0} = [x_1, x_2]$  and note  $\bar{C}_0^{i_0} \cap (\mathbb{R} \times \{x\} \times T^2) = \emptyset$  for  $x < x_1$  by Lemma 3.5 and for  $x = x_1$  since  $\bar{C}_0^{i_0}$  intersects any trivial cylinder at  $x_1$  transversely. Hence,  $\bar{C}_0^{i_0} \cap (\{0\} \times [0, x_1 + \epsilon] \times T^2)$  is empty for some  $\epsilon > 0$ . Since a trivial cylinder at  $x < x_1 + \epsilon$  with  $v_a(x) \times \sigma_{\bar{\mathcal{R}}^{i_0}}(x) = 1$  intersects each component of  $\bar{C}_0^{i_0} \cap (\mathbb{R} \times [0, x_1 + \epsilon] \times T^2)$  at least once by (9),  $\bar{C}_0^{i_0}$  cannot have both a positive and a negative end at  $x_1$ , and similarly at  $x_2$ . The claim follows since  $m^l(\bar{\mathcal{R}}_0^{i_0}) = 2$  and  $\bar{C}_0^{i_0}$  contains no trivial cylinders on  $(x_1, x_2)$ .

*Step 3.* By Lemma 3.10,  $\bar{C}_0^{i_1}$  has a negative end at some  $\bar{\rho}_{x_1}^{m_1}(\theta^-)$  with  $\theta^- \neq \theta_{x_1}^{\max}$ . Since  $\bar{C}_0^{i_2}$  must have a positive end at  $x_1$ ,  $x_1 \notin \partial(\text{supp } \mathcal{R}_{\alpha,\beta})$  by positivity of  $\bar{\mathcal{R}}^{i_1}$  and

$\bar{\mathcal{R}}^{i_2}$  at  $x_1$ . By Lemma 3.3(b),  $\sigma_{\mathcal{R}_{\alpha,\beta}}(x_1) = \sigma_{\bar{\mathcal{R}}^{i_1}}(x_1) + \sigma_{\bar{\mathcal{R}}^{i_2}}(x_1)$  is not a multiple of  $v_a(x_1)$ , so  $x_1 \in \text{int}(\text{supp } \bar{\mathcal{R}}^{i_0})$  for  $i_0 = i_1$  or  $i_2$ . In either case,  $m_{\bar{\mathcal{P}}^{i_1}}(x_1) > m_{\bar{\mathcal{R}}^{i_0}'}^{\text{triv}}(x_1) = m_{\bar{\mathcal{P}}^{i_1}}(x_1) - m_{\bar{\mathcal{R}}^{i_1}, \bar{\mathcal{R}}^{i_2}}^s(x_1)$  for  $i_0' = i_1 + i_2 - i_0$ . Hence,  $x_1 = x_0$  and the total multiplicity of trivial cylinders of  $\bar{C}^{i_0'}$  at  $x_0$  is  $m_{\bar{\mathcal{P}}^{i_1}}(x_0) - 1$  by Step 2. We conclude  $m_1 = 1$  and any other negative end of  $\bar{C}^{i_1}$  is at some  $\bar{\rho}_x^m(\theta_x^{\max})$ . Similarly,  $\bar{C}^{i_2}$  has a single positive end at  $\bar{\rho}_{x_0}(\theta^+)$  with  $\theta^+ \neq \theta_{x_0}^{\min}$  and any other positive end at some  $\bar{\rho}_x^m(\theta_x^{\min})$ .

*Step 4.* Suppose  $\bar{\rho}_{x_0}$  is convex. By Step 2, 3 and  $m^l(\bar{\mathcal{R}}_0^{i_1}) = 2$ ,

$$\Theta(\bar{C}_0^{i_1}) = 2\theta_h - \theta^- + (m(\bar{\mathcal{R}}_0^{i_1}) - 3)\theta_h/N = 0.$$

Hence,  $2\theta_h \leq \theta^- < 3\theta_h$ . Similarly,  $-2\theta_h < \theta^+ \leq -\theta_h$  by

$$\Theta(\bar{C}_0^{i_2}) = \theta_h + \theta^+ + (m(\bar{\mathcal{R}}_0^{i_2}) - 2)\theta_h/N = 0.$$

However, there is no flow of  $f_{x_0}$  from  $\theta^-$  to  $\theta^+$ . If  $\bar{\rho}_{x_0}$  is concave, we have  $\theta_h < \theta^- < 2\theta_h$  and  $-3\theta_h < \theta^+ \leq -2\theta_h$  and arrive at a similar contradiction.  $\square$

### 3.5 Invariance of the moduli count

We establish invariance of moduli count under certain deformations of  $(\lambda, J)$ .

**Definition 3.12.** Let  $\bar{\mathcal{R}}$  be a non-local indecomposable region with  $\text{supp}(\bar{\mathcal{R}}) = [x_1, x_2]$ . We say that an orbital moment map  $a$  is  $\bar{\mathcal{R}}$ -adapted if  $\bar{\mathcal{R}}$  is  $a$ -positive,  $a(x) \times a'(x_1) > 0$  for  $x < x_1$  and  $a(x) \times a'(x_2) > 0$  for  $x > x_2$ .

**Lemma 3.13.** For a non-local indecomposable region  $\bar{\mathcal{R}}$ , the space of  $\bar{\mathcal{R}}$ -adapted orbital moment maps is path-connected.

*Proof.* Observe that a *rescaling*  $\kappa a$  of an orbital moment map  $a$  by  $\kappa : \mathbb{I} \rightarrow \mathbb{R}^+$  and a *reparametrization*  $\psi^*a$  by  $\psi \in \text{aut}(\mathbb{I})$  are connected to  $a$  via linear interpolations of  $\kappa$  with  $\kappa' \equiv 1$  and  $\psi$  with  $\text{id}$ , respectively.

Let  $a$  be any  $\bar{\mathcal{R}}$ -adapted orbital moment map and let  $P = \{x_1 < \dots < x_n\}$  be  $\text{supp}(m_{\bar{\mathcal{R}}})$  if  $m(\bar{\mathcal{R}}) > 2$ ; and  $\text{supp}(m_{\bar{\mathcal{R}}}) \cup \{x_2\}$  for  $x_2 \in \text{int}(\text{supp } \bar{\mathcal{R}})$  otherwise. In either case, we can find  $u_i$  near  $-\sigma_{\bar{\mathcal{R}}}(x_{i+1})^\vee$  so that  $u_i \times a' > 0$  on  $[x_i, x_{i+1}]$  for each  $1 \leq i < n$ . We claim there exist  $\{z_i\}_{i=1}^{n-1}$  and a continuous rescaling  $\bar{b}$  of  $a$  such that  $z_i \in (x_i, x_{i+1})$ ,  $\bar{b}$  is smooth except (possibly) at  $z_i$  and  $\bar{b}'(x_i) \sim a'(x_i)$  for each  $i$ : if  $u_i \times a(x_i) \geq 0$ , pick  $z_i$  close to  $x_i$ ; if  $u_i \times a(x_{i+1}) \leq 0$ , pick  $z_i$  close to  $x_{i+1}$ ; if neither, pick  $z_i$  so that  $a(z_i) \sim u_i$ . The claim follows from  $a \times a' > 0$  and  $u_i \times a' > 0$ . Obtain a smooth rescaling  $b$  of  $\bar{b}$  by modifying  $\bar{b}$  on small neighbourhoods of  $z_i$  so that  $b \times b'$  does not change signs on  $[x_i, x_{i+1}]$ .

For  $\bar{\mathcal{R}}$ -adapted  $a_0$  and  $a_1$ , find  $b_0$  and  $b_1$  using the above procedure. By rescaling, assume  $a_0$  and  $a_1$  are  $C^1$ -close to  $b_0$  and  $b_1$  and by reparametrizing,  $b_0'(x) \sim b_1'(x)$



for all  $x$ . Then,  $a_r := (1-r)a_0 + ra_1$  is an orbital moment map since  $b_r \times b'_r > 0$  for  $b_r := (1-r)b_0 + rb_1$ , and it is  $\bar{\mathcal{R}}$ -adapted since  $a_0$  and  $a_1$  are.  $\square$

The following is an adaptation of [10, Lemma 3.15]:

**Proposition 3.14.** Let  $\mathcal{R}$  be an indecomposable region with  $I(\mathcal{R}) = 1$  and  $\bar{I}(\mathcal{R}) \leq 0$ . For each  $r \in \{0, 1\}$ , let  $a_r$  be an  $\mathcal{R}$ -adapted orbital moment map,  $\lambda_r$  a good perturbation of  $\pi_{\mathbb{I}}^* a_r$  and  $J_r$  a generic  $\lambda_r$ -admissible almost complex structure. Then

$$\#(\mathcal{M}_1^{J_0}(\alpha_0, \beta_0)/\mathbb{R}) = \#(\mathcal{M}_1^{J_1}(\alpha_1, \beta_1)/\mathbb{R})$$

where  $\alpha_r$  and  $\beta_r$  are orbit sets of  $\lambda_r$  with  $\mathcal{R}_{\alpha_r, \beta_r} = \mathcal{R}$ .

*Proof.* Suppose we have paths  $\{a_r\}_{r \in [0, 1]}$  of  $\mathcal{R}$ -adapted orbital moment maps with  $\mathcal{A}^r := \sum_x m_{\mathcal{R}}(x) \mathcal{A}_{a_r}(x)$ ; and  $\{\lambda_r\}$  of perturbations of  $\pi_{\mathbb{I}}^* a_r$  so that, for each  $r \in [0, 1]$  and  $x \in \text{supp}(m_{\mathcal{R}})$ ,  $\lambda_r|_{U_x^r \times T^2}$  is of the form (7) for a neighbourhood  $U_x^r$  of  $x$  with  $\mathcal{A}_{a_r}|_{U_x^r \setminus \{x\}} > \mathcal{A}^r$ . Let  $\alpha_r$  and  $\beta_r$  denote the orbit sets of  $\lambda_r$  with  $\mathcal{R}_{\alpha_r, \beta_r} = \mathcal{R}$  and choose a generic path  $\{J_r\}$  of  $\lambda_r$ -admissible almost complex structures. Assuming compactness of  $\mathcal{M}_1^{J_r}(\alpha_r, \beta_r)/\mathbb{R}$  as in [9], its mod 2 count can change only when there is a broken  $J_r$ -holomorphic curve  $C = (C^1, \dots, C^l)$  from  $\alpha_r$  to  $\beta_r$ .

Due to action, a  $d$ -fold cover  $dC'$  of a somewhere injective  $J_r$ -holomorphic curve  $C' \subset \mathbb{R} \times U_x^r \times T^2$  has  $I(dC') = dI(C') \geq 0$  with equality only if  $C'$  is trivial. This and Lemma 3.3 implies each  $I(C^i) \geq 0$  and some  $C^i$  must contain a somewhere injective component  $C' \subset \mathbb{R} \times U_x^r \times T^2$  with  $I(C') = 1$ . If  $\mathcal{R}$  is local,  $C = C'$  and we are done. Otherwise, by Proposition 3.11 and the local case of this Proposition, such  $C'$  exists in pairs and by standard gluing results as in [13],  $C$  also exists in pairs (cf. automatic transversality [18]). It remains to provide such paths and justify compactness.

Write  $\lambda_r = (1 + \eta_r \sum \tilde{f}_{x^r}^r) \pi_{\mathbb{I}}^* a_r$  for  $r = 0, 1$ . If  $a_0 = a_1$ , linearly interpolating  $\sum \tilde{f}_{x^r}^r$  and choosing sufficiently small  $\eta_r$  gives a path of good perturbations of  $\pi_{\mathbb{I}}^* a_0$  and Gromov compactness holds. In general, let  $\{a_r\}$  be given by a linear interpolation if  $\mathcal{R}$  is local, and by Lemma 3.13 otherwise. For each  $x \in P := \text{supp}(m_{\mathcal{R}})$ , the first step allows us to assume  $\tilde{f}_x^0 = \tilde{f}_x^1$  and  $U_x^r := U_x^0$  is small enough so that  $\lambda_r := (1 + \sum_{x \in P} \eta_r \tilde{f}_x^0) \pi_{\mathbb{I}}^* a_r$  gives a desired path for small enough  $\eta_r$ .

Finally, we justify Gromov compactness when  $\mathcal{R}$  is not local. Let  $C \in \mathcal{M}_1^{J_r}(\alpha_r, \beta_r)$  and  $x_0 \in P$ . We claim  $\pi|_C^{-1}(U_{x_0}^0)$  is connected where  $\pi : \mathbb{R} \times Y \rightarrow \mathbb{I}$  is the obvious projection. For each of its component  $C'$ ,  $v_{\mathcal{R}}(x_0) \times [\pi|_{C'}^{-1}(x)] \geq 0$  for generic  $x \in U_{x_0}^0$  by (9). If this is equality for some  $C'$ , it has a positive end at a convex orbit or a negative end at a concave orbit (cf. proof of Lemma 3.3(b)) so  $x_0 \in \partial(\text{supp } \mathcal{R})$  and the claim follows from  $m_{\mathcal{R}}(x_0) = 1$ . Otherwise, it follows from  $v_{\mathcal{R}}(x_0) \times \sigma_{\mathcal{R}}(x) \leq 1$ . In turn,  $\pi|_C^{-1}(V)$  for any component  $V$  of  $\text{supp}(\mathcal{R}) \setminus \cup_{x \in P} U_x^0$  is a single finite cylinder since  $C$  does not have genus by (2) and any component of  $\pi|_C^{-1}(\partial V)$  is homologically

non-trivial by (9). Since  $\lambda_r$  does not have an orbit  $\rho$  in  $V$  with  $[\rho] = \pm\sigma_{\mathcal{R}}(x)$  for  $x \in V$ , a sequence  $C_n \in \mathcal{M}_1^{J_{r_n}}(\alpha_{r_n}, \beta_{r_n})$  cannot develop an end in  $V$ . (When there is a birth/death of such  $\rho$ , this assertion fails and a bifurcation does occur.)  $\square$

### 3.6 Base cases for induction

Let  $\lambda$  be a good perturbation of  $\pi_{\mathbb{I}}^*a$  for a generic orbital moment map  $a$  and  $J$  a generic  $\lambda$ -admissible almost complex structure.

**Proposition 3.15.** Let  $\alpha$  and  $\beta$  be orbit sets of  $\lambda$ . Suppose  $\mathcal{R}_{\alpha,\beta}$  is  $a$ -positive,  $I(\mathcal{R}_{\alpha,\beta}) = 1$  and one of the following holds:

- (i)  $m_{\mathcal{P}_{\alpha}} = \delta_{x_+}$  and  $m_{\mathcal{P}_{\beta}} = \delta_{x_-}$  for  $\delta_+ \neq \delta_-$ ;
- (ii)  $c_{\mathcal{R}} \geq 0$ ,  $m_{\mathcal{P}_{\alpha}} = \delta_{x_1} + \delta_{x_2}$ ,  $m_{\mathcal{P}_{\beta}} = m\delta_{x_0}$  for  $x_0 \in (x_1, x_2)$ , and  $m_{\mathcal{P}_{\beta}}^h(x_0) = 0$  if  $m > 1$ ; or
- (iii)  $c_{\mathcal{R}} \leq 0$ ,  $m_{\mathcal{P}_{\beta}} = \delta_{x_1} + \delta_{x_2}$ ,  $m_{\mathcal{P}_{\alpha}} = m\delta_{x_0}$  for  $x_0 \in (x_1, x_2)$ , and  $m_{\mathcal{P}_{\alpha}}^h(x_0) = 0$  if  $m > 1$ .

Then  $\#(\mathcal{M}_1(\alpha, \beta)/\mathbb{R}) = 1$ .

*Proof.* Since  $\bar{I}(\mathcal{R}_{\alpha,\beta}) = 0$ , we may assume  $a$  is any  $\mathcal{R}_{\alpha,\beta}$ -adapted orbital moment map and  $\lambda$  any good perturbation by Proposition 3.14. With this assumption, cases (ii) and (iii) are covered in [10, 11] but we refer to their original source in [17].

*Case (i)* Assume  $x_{\pm}$  are critical points of  $g(x) := \pm x(x - 1/2)(x - 1)$  with a plus sign if  $x_+ < x_-$  and minus otherwise. By  $SL_2(\mathbb{Z})$ -symmetry, assume  $v(x_+) = (1, 1)$  and  $\lambda = (1 + \eta f)(1 + \eta' g)\pi_{\mathbb{I}}^*a_{\lambda_{std}}$  for a small  $\eta' > 0$  where  $a_{\lambda_{std}}(x) = (1 - x, x)$  is the orbital moment map of  $(S^3, \lambda_{std})$  and  $f := \sum \tilde{f}_x$  in (7). Finite dimensional Morse-Bott theory on  $S^2$ -family of embedded orbits of  $\lambda_{std}$  gives a unique flow of  $(1 + \eta f)(1 + \eta' g)$  from  $\bar{\rho}_{x_+}(\theta_e)$  to  $\bar{\rho}_{x_-}(\theta_h)$ . By a Morse-Bott argument [1], the  $J$ -holomorphic cylinder from  $\check{e}_{x_+}$  to  $\hat{h}_{x_-}$  corresponds to this flow for small  $\eta$  and  $\eta'$ . We similarly get the unique member of  $\mathcal{M}_1(\check{h}_{x_+}, \hat{e}_{x_-})/\mathbb{R}$ .

*Case (ii)* There is an identification of  $\mathbb{R} \times (\mathbb{I} \times S^1 \times S^1)$  with a subset of  $(\mathbb{R} \times S^2 \times S^1, \alpha, J)$  considered in [17] so that the pullback of  $\alpha$  is  $\pi_{\mathbb{I}}^*a$  for an everywhere convex  $\mathcal{R}_{\alpha,\beta}$ -adapted  $a$  and the pullback of  $J$  is  $\pi_{\mathbb{I}}^*a$ -admissible. The unique member of  $\mathcal{M}_1(\alpha, \beta)/\mathbb{R}$  for  $m = 0$  corresponds to an  $(\mathbb{R} \times S^1)$ -family of  $J$ -holomorphic cylinders in [17, Theorem A.1(c)] via a Morse-Bott argument [1]. If  $m > 0$ ,  $C \in \mathcal{M}_1(\alpha, \beta)$  has one negative puncture by partition condition and we similarly get the unique member of  $\mathcal{M}_1(\alpha, \beta)/\mathbb{R}$  from an  $(\mathbb{R} \times S^1 \times S^1)$ -family of three-punctured  $J$ -holomorphic spheres in [17, Theorem A.2].

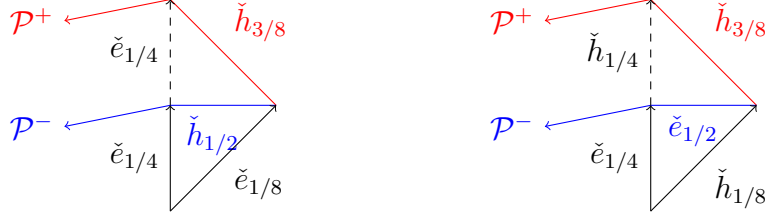


Figure 3: Case 1 of the induction step.

*Case (iii)* We reduce to case (ii) with  $m > 0$ . Assume  $a$  is  $\mathcal{R}_{\alpha,\beta}$ -adapted and everywhere concave and  $rdt_1 \in \text{im}(a)$  for some  $r \in \mathbb{R}$ . Let  $\psi(t_1, t_2) = (t_1, -t_2)$  and  $\Psi = (-\text{id}_{\mathbb{R}}) \times \text{id}_{\mathbb{I}} \times \psi$  be diffeomorphisms of  $T^2$  and  $\mathbb{R} \times \mathbb{I} \times T^2$ . Then  $\lambda^\dagger = 2rdt_1 - (\text{id}_{\mathbb{I}} \times \psi)^* \lambda$  is a good perturbation of an everywhere convex orbital moment map and, for each orbit set  $\gamma$  of  $\lambda$ ,  $\gamma^\dagger = (\text{id}_{\mathbb{I}} \times \psi)_*^{-1} \gamma$  is an orbit set of  $\lambda^\dagger$ . We have  $\mathcal{M}_1^J(\alpha, \beta) = \mathcal{M}_1^{J^\dagger}(\beta^\dagger, \alpha^\dagger)$  where  $J$  is a  $\lambda$ -admissible almost complex structure and  $J^\dagger := \Psi_*^{-1} J \Psi_*$ . Furthermore, if  $J$  is close to  $\pi_{\mathbb{I}}^* a$ -admissible  $\bar{J}$  with  $\bar{J}(\partial_x) = c\bar{Q}$ , then  $J^\dagger$  maps  $\partial_s$  to a positive multiple of the Reeb vector field of  $\lambda^\dagger$  and is  $d(e^s \lambda^\dagger)$ -tame provided  $c > 0$  is small. We can deform  $J^\dagger$  to a  $\lambda^\dagger$ -admissible almost complex structure without changing the moduli count, similarly to Proposition 3.14.  $\square$

### 3.7 Induction step

**Proposition 3.16.** Let  $(Y, \lambda, J)$  be as in §3.6 and let  $\alpha$  and  $\beta$  be orbit sets of  $\lambda$ . Suppose  $\mathcal{R}_{\alpha,\beta}$  is indecomposable and  $a$ -positive with  $I(\mathcal{R}_{\alpha,\beta}) = 1$  and  $\bar{I}(\mathcal{R}_{\alpha,\beta}) = 0$ . Then  $\#(\mathcal{M}_1(\alpha, \beta)/\mathbb{R}) = 1$ .

*Proof.* Since  $\mathcal{R}_{\alpha,\beta}$  is minimally positive, write  $\text{supp } \mathcal{P}_\alpha \sqcup \text{supp } \mathcal{P}_\beta = \{x_1 < \dots < x_n\}$ . We define the “induction complexity”  $z(\mathcal{R}_{\alpha,\beta})$  as follows: if  $\mathcal{R}_{\alpha,\beta}$  is of the form in Proposition 3.15, set  $z(\mathcal{R}_{\alpha,\beta}) := 1$ . If  $n = 3$  with  $m_{\mathcal{P}_\alpha}(x_1) = m_{\mathcal{P}_\alpha}(x_3) = m_{\mathcal{P}_\beta}^e(x_2) = m_{\mathcal{P}_\beta}^h(x_2) = 1$  or  $m_{\mathcal{P}_\beta}(x_1) = m_{\mathcal{P}_\beta}(x_3) = m_{\mathcal{P}_\alpha}^e(x_2) = m_{\mathcal{P}_\alpha}^h(x_2) = 1$ , set  $z(\mathcal{R}_{\alpha,\beta}) := 2$ . Otherwise,  $z(\mathcal{R}_{\alpha,\beta}) := m(\mathcal{R}_{\alpha,\beta}) \geq 3$ . We induct on  $z$ .

Assume  $x_1 \in \text{supp } \mathcal{P}_\alpha$  as the case with  $x_1 \in \text{supp } \mathcal{P}_\beta$  can be argued similarly by reversing the roles of  $\mathcal{P}_\alpha$  and  $\mathcal{P}_\beta$ . Further assume  $m_{\mathcal{P}_\alpha}^h(x_1) = 1$  as the case with  $m_{\mathcal{P}_\alpha}^e(x_1) = 1$  can be argued similarly by switching the roles of  $x_i$  and  $x_{n-i}$ . Note that  $\bar{I}(\mathcal{R}_{\alpha,\beta}) = 0$  ensures admissibility of  $\alpha$  and  $\beta$  and that  $Q_{\mathcal{R}_{\alpha,\beta}}(x_2) \neq 0$  since we assume  $z(\mathcal{R}_{\alpha,\beta}) > 1$ .

*Case 1.* If  $x_2 \in \text{supp } \mathcal{P}_\beta$ , assume  $v_{\mathcal{P}_\alpha}(x_1) = (-1, 1)$  and  $v_{\mathcal{P}_\beta}(x_2) = (-1, 0)$  by  $SL_2(\mathbb{Z})$ -symmetry and assume  $(a')^\vee|_{[0, 1/2+\epsilon)} \sim (\cos 2\pi x, \sin 2\pi x)$  using Proposition 3.14 and  $\text{aut}(\mathbb{I})$ . Write  $\mathcal{P}_\alpha = \mathcal{P}_{\check{h}_{3/8}} \mathcal{P}^+$  and  $\mathcal{P}_\beta = \mathcal{P}_{1/2} \mathcal{P}^-$  where  $\mathcal{P}_{1/2} = \mathcal{P}_{\check{h}_{1/2}}$  if  $m_{\mathcal{P}_\beta}^h(1/2) > 0$  and  $\mathcal{P}_{\check{e}_{1/2}}$  otherwise. Let  $\tilde{\alpha}$  and  $\tilde{\beta}$  be orbit sets so that  $\mathcal{P}_{\tilde{\alpha}} = \mathcal{P}_{1/8} \mathcal{P}_\alpha$



Figure 4: Case 2 of the induction step.

and  $\mathcal{P}_{\tilde{\beta}} = \mathcal{P}_{\tilde{e}_{1/4}} \mathcal{P}^-$  where  $\mathcal{P}_{1/8} = \mathcal{P}_{\tilde{e}_{1/8}}$  if  $m_{\mathcal{P}_{\beta}}^h(1/2) > 0$  and  $\mathcal{P}_{\tilde{h}_{1/8}}$  otherwise. See Figure 3. We consider  $\gamma$  for which  $\langle \partial \tilde{\alpha}, \gamma \rangle \langle \partial \gamma, \tilde{\beta} \rangle \neq 0$ .

Each of  $\mathcal{R}^1 = (\mathcal{P}_{\tilde{\alpha}}, \mathcal{P}_{\gamma})$  and  $\mathcal{R}^2 = (\mathcal{P}_{\gamma}, \mathcal{P}_{\tilde{\beta}})$  contains exactly one non-local factor  $\mathcal{R}_0^i$  and  $\bar{I}(\mathcal{R}_0^i) = 0$  by Lemma 3.9 since  $\bar{I}(\mathcal{P}_{\tilde{\alpha}}, \mathcal{P}_{\tilde{\beta}}) = 1$ . By positivity of  $\mathcal{R}^i$ ,  $\min(\text{supp } \mathcal{P}_{\gamma}) = 1/8$  or  $1/4$ . In the first case, write  $\text{supp}(\mathcal{R}_0^2) = [1/8, x_0]$  where  $x_0 \geq 1/2$  since  $\mathcal{P}_{\tilde{\beta}}|_{(1/4, 1/2)} \equiv 0$ . By minimal positivity of  $\mathcal{R}_0^2$ ,  $\mathcal{P}_{\gamma}|_{(1/8, 1/2)} \equiv 0$  and by  $a$ -positivity of  $\mathcal{R}_0^2$  at  $x = 1/2$ ,  $x_0$  must be  $1/2$ . Since  $I(\mathcal{R}_0^2) = 1$ , we have  $\mathcal{P}_{\gamma} = \mathcal{P}_{1/8} \mathcal{P}_{\beta}$  and  $\langle \partial \tilde{\alpha}, \gamma \rangle \langle \partial \gamma, \tilde{\beta} \rangle = \langle \partial \alpha, \beta \rangle$  by induction hypothesis. In the latter case,  $\text{supp}(\mathcal{R}_0^1) = [1/8, 3/8]$  by minimal positivity of  $\mathcal{R}_0^1$  and  $\mathcal{P}_{\gamma} = \mathcal{P}_{\tilde{e}_{1/4}} \mathcal{P}_{\tilde{e}_{1/4}} \mathcal{P}^+$  or  $\mathcal{P}_{\tilde{e}_{1/4}} \mathcal{P}_{\tilde{h}_{1/4}} \mathcal{P}^+$ , whichever makes  $I(\mathcal{R}_0^1) = 1$ . We verify that  $z(\mathcal{R}_0^i) < z(\mathcal{R}_{\alpha, \beta})$  for each  $i$ : if  $z(\mathcal{R}_{\alpha, \beta}) = 2$ ,  $z(\mathcal{R}_0^i) = 1$ ; otherwise, it follows from  $m(\mathcal{R}_0^2) = m(\mathcal{R}_{\alpha, \beta}) - 1$ . By induction hypothesis and  $\partial^2 = 0$ , we conclude  $\langle \partial \alpha, \beta \rangle = 1$ .

*Case 2.* If  $x_2 \in \text{supp } \mathcal{P}_{\alpha}$ , assume  $v_{\mathcal{P}_{\alpha}}(x_1) = (0, 1)$ ,  $v_{\mathcal{P}_{\alpha}}(x_2) = (-1, 0)$  for  $x_2 = 1/2 + 2\epsilon$  and  $(a')^{\vee}|_{[0, 1/2 + \epsilon]} \sim (\cos 2\pi x, \sin 2\pi x)$  with  $\mathcal{A}_a|_{(1/2, x_2)} > L$ . Write  $\mathcal{P}_{\alpha} = \mathcal{P}_{\tilde{h}_{1/4}} \mathcal{P}_{x_2} \mathcal{P}^+$  where  $\mathcal{P}_{x_2} = \mathcal{P}_{\tilde{h}_{x_2}}$  if  $m_{\mathcal{P}_{\alpha}}^h(x_2) > 0$  and  $\mathcal{P}_{\tilde{e}_{x_2}}$  otherwise. Let  $\tilde{\alpha}$  be an orbit set so that  $\mathcal{P}_{\tilde{\alpha}} = \mathcal{P}_{\tilde{h}_{1/4}} \mathcal{P}_{1/2} \mathcal{P}^+$  where  $\mathcal{P}_{1/2} = \mathcal{P}_{\tilde{e}_{1/2}}$  if  $m_{\mathcal{P}_{\alpha}}^h(x_2) > 0$  and  $\mathcal{P}_{\tilde{h}_{1/2}}$  otherwise. See Figure 4.

As before, if  $\langle \partial \tilde{\alpha}, \gamma \rangle \langle \partial \gamma, \beta \rangle \neq 0$ , each of  $\mathcal{R}^1 = (\mathcal{P}_{\tilde{\alpha}}, \mathcal{P}_{\gamma})$  and  $\mathcal{R}^2 = (\mathcal{P}_{\gamma}, \mathcal{P}_{\beta})$  contains exactly one non-local factor  $\mathcal{R}_0^i$  and  $\bar{I}(\mathcal{R}_0^i) = 0$ . If  $\min(\text{supp } \mathcal{P}_{\gamma}) = 1/4$ ,  $m_{\mathcal{P}_{\gamma}}(1/2) = 0$  by minimal positivity of  $\mathcal{R}_0^2$  and  $m_{\mathcal{P}_{\gamma}}(x_2) = 1$  by  $a$ -positivity of  $\mathcal{R}_0^1$ , yielding  $\langle \partial \tilde{\alpha}, \gamma \rangle \langle \partial \gamma, \beta \rangle = \langle \partial \alpha, \beta \rangle$  by induction hypothesis. If  $\min(\text{supp } \mathcal{P}_{\gamma}) > 1/4$ ,  $\text{supp}(\mathcal{R}_0^1) = [1/4, 1/2]$ , so  $\mathcal{P}_{\gamma} = \mathcal{P}_{\tilde{e}_{3/8}} \mathcal{P}^+$  or  $\mathcal{P}_{\tilde{h}_{3/8}} \mathcal{P}^+$ , whichever makes  $I(\mathcal{R}_0^1) = 1$ . Since  $z(\mathcal{R}_0^i) < z(\mathcal{R}_{\alpha, \beta})$ ,  $\langle \partial \alpha, \beta \rangle = 1$  by induction hypothesis and  $\partial^2 = 0$ .  $\square$

## 4 ECC of closed manifolds

### 4.1 Toric contact $T^3$

Define  $s_n : \mathbb{R} \rightarrow \mathbb{R}$  by  $s_n(x) := x - n$  and let  $q : \mathbb{R} \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$  be the quotient map. Previous definitions regarding lattice paths and regions still make sense when

we replace  $\mathbb{I}$  with  $\mathbb{R}$  or  $S^1 = \mathbb{R}/\mathbb{Z}$  but we need to supplement Definition 1.4 with:

**Definition 4.1.** An *offset (lattice) region*  $\bar{\mathcal{R}}$  is a triple  $(\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^1, \sigma_0)$  where  $\bar{\mathcal{P}}^i : S^1 \rightarrow \bar{\mathcal{V}}$  are paths with  $[\bar{\mathcal{P}}^0] = [\bar{\mathcal{P}}^1]$  and  $\sigma_0 \in \Lambda$ . The *slice class* of  $\bar{\mathcal{R}}$  at  $x_0 \in [0, 1) \stackrel{\text{set}}{=} S^1$  is

$$\sigma_{\bar{\mathcal{R}}}(x_0) := \sigma_0 - \sum_{x \in [0, x_0)} m_{\bar{\mathcal{P}}^0}(x) \cdot v_{\bar{\mathcal{P}}^0}(x) + \sum_{x \in [0, x_0)} m_{\bar{\mathcal{P}}^1}(x) \cdot v_{\bar{\mathcal{P}}^1}(x) \in \Lambda.$$

We say  $\bar{\mathcal{R}}_{\mathbb{R}} = (\bar{\mathcal{P}}_{\mathbb{R}}^0, \bar{\mathcal{P}}_{\mathbb{R}}^1)$  with  $\bar{\mathcal{P}}_{\mathbb{R}}^i : \mathbb{R} \rightarrow \bar{\mathcal{V}}$  is a *lift* of  $\bar{\mathcal{R}}$  if  $\cup_{n \in \mathbb{Z}} S_n^*(\bar{\mathcal{P}}_{\mathbb{R}}^i) = q^* \bar{\mathcal{P}}^i$  and  $\sum_{n \in \mathbb{Z}} \sigma_{\bar{\mathcal{R}}_{\mathbb{R}}}(n) = \sigma_0$ . We similarly define a lift of a decorated  $\mathcal{R}$ . A *lift* of a pair  $(\mathcal{P}^0, \mathcal{P}^1)$  is a lift of  $(\mathcal{P}^0, \mathcal{P}^1, \sigma)$  for some  $\sigma \in \Lambda$ . We say  $\bar{\mathcal{R}}$  is *decomposable* if it lifts to a decomposable region.

In this section, a region will always mean an offset region unless we refer to a lift, in which case we use the subscript  $\mathbb{R}$ . A concatenation  $\bar{\mathcal{R}}_1 \bar{\mathcal{R}}_2$  still makes sense if  $\text{int}(\text{supp } \bar{\mathcal{R}}_i)$  are disjoint, and so does factoring  $\bar{\mathcal{R}}$  into indecomposable  $\bar{\mathcal{R}}_k$  up to cyclic ordering.

Now consider a generic orbital moment map  $a : S^1 \rightarrow (\mathfrak{t}^2)^*$ . Define  $\varphi_a, \varphi_{a'} : \mathbb{R} \rightarrow \mathbb{R}$  so that  $a^\vee \sim (\cos \varphi_a, \sin \varphi_a)$ ,  $(a')^\vee \sim (\cos \varphi_{a'}, \sin \varphi_{a'})$ ,  $\varphi_a(0) \in [0, 2\pi)$  and  $\varphi_{a'} - \varphi_a \in (0, \pi)$ . By  $a \times a' > 0$ ,  $\varphi_a(1) = \varphi_a(0) + 2\pi n_a$  for  $n_a \geq 1$  and  $\varphi_{a'}(1) = \varphi_{a'}(0) + 2\pi n_{a'}$ . Equip  $Y = S^1 \times T^2$  with a small (in the sense of Proposition 3.11) good perturbation  $\lambda$  of  $\pi_{S^1}^* a$  and  $\mathbb{R} \times Y$  with a generic  $\lambda$ -admissible almost complex structure  $J$ . For orbit sets  $\alpha$  and  $\beta$  of  $\lambda$ , define

$$H_2(Y, \alpha, \beta, \sigma) := H_2(Y, \alpha, \beta, 0) + [S^1] \times \sigma$$

for  $H_2(Y, \alpha, \beta, 0) := (q \times \text{id}_{T^2})_* H_2([0, 1) \times T^2, \alpha, \beta)$  and  $\sigma \in H_1(\{pt\} \times T^2)$ ,

$$\mathcal{M}(\alpha, \beta, \sigma) := \{\mathcal{C} \in \mathcal{M}(\alpha, \beta) \mid [\mathcal{C}] \in H_2(Y, \alpha, \beta, \sigma)\}$$

and  $\mathcal{M}_k(\alpha, \beta, \sigma) := \{\mathcal{C} \in \mathcal{M}(\alpha, \beta, \sigma) \mid I(\mathcal{C}) = k\}$ .

**Proposition 4.2.** We adapt some previous results to this setting. The proofs remain nearly identical and uses the same trivialization  $\tau$  of  $\xi \cong \text{span}\{\partial_x, \bar{Q}\}$ .

(a) Proposition 3.2 now asserts  $I(\mathcal{R}_{\alpha, \beta, \sigma}) = I(\alpha, \beta, Z)$  for  $Z \in H_2(Y, \alpha, \beta, \sigma)$ : if  $\mathcal{R}_{\mathbb{R}}$  is a lift of  $\mathcal{R}_{\alpha, \beta, 0}$  with  $\text{supp}(\mathcal{R}_{\mathbb{R}}) \subset [0, 1)$ ,  $I(\mathcal{R}_{\alpha, \beta, 0}) = I(\mathcal{R}_{\mathbb{R}})$ . Otherwise, use (3) and

$$I(\mathcal{P}^0, \mathcal{P}^1, \sigma_1) - I(\mathcal{P}^0, \mathcal{P}^1, \sigma_0) = 2[\mathcal{P}^0] \times (\sigma_1 - \sigma_0).$$

(b) Lemma 3.3 (a) works unmodified while part (b) says: if  $\bar{\mathcal{R}}$  is  $a$ -positive, then it is positive and, whenever  $a'(x)^\vee \times \sigma_{\bar{\mathcal{R}}}(x) = 0$ , it lifts to  $q^* a$ -positive  $\bar{\mathcal{R}}_{\mathbb{R}}$  with  $\text{supp}(\bar{\mathcal{R}}_{\mathbb{R}}) \subset [x, x+1]$ . Lemma 3.5 says:  $\mathcal{R}_{\alpha, \beta, \sigma}$  is  $a$ -positive if  $\mathcal{M}(\alpha, \beta, \sigma) \neq \emptyset$ , and the second assertion still holds.

(c) Lemma 3.9 works for  $\bar{\mathcal{R}} = (\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^2, \sigma^1 + \sigma^2)$ ,  $\bar{\mathcal{R}}^1 = (\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^1, \sigma^1)$  and  $\bar{\mathcal{R}}^2 = (\bar{\mathcal{P}}^1, \bar{\mathcal{P}}^2, \sigma^2)$ . Lemma 3.10 works unmodified.

**Definition 4.3.** We say a lift  $\mathcal{R}_{\mathbb{R}}$  is *relevant* if it satisfies the criteria in Theorem 1.1 and the normalizing conditions (i)  $\text{int}(\text{supp } \mathcal{R}_{\mathbb{R}}) = (x_1, x_2)$  for  $x_1 \in [0, 1)$  and  $x_2 \in [x_1, x_1 + 1]$  and (ii)  $\text{supp}(m_{\mathcal{R}_{\mathbb{R}}}^{\text{triv}}) \subset [x_1, x_1 + 1)$ .

**Theorem 4.4.** Consider  $(Y, \lambda, J)$  as above. For admissible orbit sets  $\alpha$  and  $\beta$  of  $\lambda$ ,  $\langle \partial\alpha, \beta \rangle = 1$  if and only if  $(\mathcal{P}_{\alpha}, \mathcal{P}_{\beta})$  admits a unique relevant lift.

**Remark 4.5.** (Uniqueness of a relevant lift) Relevant lifts are possible only if we can write  $\alpha = \alpha' \cup \gamma$  and  $\beta = \beta' \cup \gamma$  with  $m^l(\mathcal{R}_{\alpha', \beta', 0}) = 2$ . Suppose  $\mathcal{R}_{\mathbb{R}}$  and  $\mathcal{R}'_{\mathbb{R}}$  are relevant lifts of  $\mathcal{R}_{\alpha, \beta, \sigma}$  and  $\mathcal{R}_{\alpha, \beta, \sigma'}$  with  $\sigma \neq \sigma'$ . Then, without loss of generality,  $\text{int}(\text{supp } \mathcal{R}_{\mathbb{R}}) = (x_1, x_2)$ ,  $\text{int}(\text{supp } \mathcal{R}'_{\mathbb{R}}) = (x_2, x_1 + 1)$  and  $m_{\mathcal{R}_{\alpha', \beta', 0}}^l = \delta_{x_1} + \delta_{x_2}$ . Since  $\mathcal{R}_{\mathbb{R}}$  is minimally positive,  $\gamma$  has no orbits on  $(x_1, x_2)$ , and since  $\sigma_{\mathcal{R}'_{\mathbb{R}}}|_{(x_1, x_2)} \equiv 0$ , neither does  $\alpha' \cup \beta'$ . Similarly,  $\alpha \cup \beta$  has no orbits on  $(x_2, x_1 + 1)$ . We conclude (i)  $\text{supp}(\mathcal{P}_{\alpha} \cup \mathcal{P}_{\beta}) = \{x_1, x_2\}$  and  $a'(x) \times a'(x_1) \neq 0$  except at  $x_1$  and  $x_2$ ; and (ii) since  $n_a \geq 1$ ,  $a'$  is convex at  $x_1$  and  $x_2$  and  $\beta' = \emptyset$ . In all other cases, we have at most one  $\sigma$  for which  $\mathcal{R}_{\alpha, \beta, \sigma}$  admits a relevant lift. Moreover, this lift is unique unless  $m_{\mathcal{R}_{\alpha', \beta', 0}}^l = 2\delta_{x_0}$ : in this case, there are two relevant lifts (with  $\text{supp}(\mathcal{R}_{\mathbb{R}}) = [x_0, x_0 + 1]$ ), one with a hyperbolic edge at  $x_0$  and the other at  $x_0 + 1$ .

We first show a basic property of (offset) regions:

**Lemma 4.6.** Let  $\bar{\mathcal{R}} = (\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^1, \sigma)$  be  $a$ -positive. If  $\bar{\mathcal{P}}^0|_{(x_1, x_2)} \equiv 0$  and  $\sigma_{\bar{\mathcal{R}}}$  does not vanish on  $(x_1, x_2)$ , then  $\varphi_{a'}(x_2) - \varphi_{a'}(x_1) \leq \pi$  with equality only if  $Q_{\bar{\mathcal{R}}} = 0$  at  $x_1$  and  $x_2$ . In particular, if  $\bar{\mathcal{R}}$  does not lift,  $m(\bar{\mathcal{P}}^0) > 2$ .

*Proof.* By  $a$ -positivity, we can choose  $\varphi_{\sigma} : (x_1, x_2) \rightarrow \mathbb{R}$  so that  $\sigma_{\bar{\mathcal{R}}} \sim (\cos \varphi_{\sigma}, \sin \varphi_{\sigma})$  and  $\varphi_{\sigma} - \varphi_{a'} \in (0, \pi)$ . The first statement follows since  $\varphi_{\sigma}$  jumps by  $\vartheta \in (-\pi, 0)$  at  $\text{supp } \bar{\mathcal{P}}^1$  and is constant everywhere else. The second follows from  $n_a \geq 1$ .  $\square$

*Proof of Theorem 4.4.* Suppose  $m^l(\mathcal{R}_{\alpha, \beta, 0}) \geq 2$ . Equip  $\tilde{Y} = \mathbb{R} \times T^2$  with a contact form  $\tilde{\lambda} := (q \times \text{id}_{T^2})^* \lambda$  and  $\mathbb{R} \times \tilde{Y}$  with an almost complex structure  $\tilde{J} := \tilde{q}_*^{-1} J \tilde{q}_*$  where  $\tilde{q} = \text{id}_{\mathbb{R}} \times q \times \text{id}_{T^2}$ . We claim  $\mathcal{M}_1(\alpha, \beta) = \cup \mathcal{M}_1^{\tilde{J}}(\tilde{\alpha}, \tilde{\beta})$  where the union is over orbit sets  $\tilde{\alpha}$  and  $\tilde{\beta}$  of  $\tilde{\lambda}$  for which  $\mathcal{R}_{\tilde{\alpha}, \tilde{\beta}}$  is a relevant lift of  $(\mathcal{P}_{\alpha}, \mathcal{P}_{\beta})$ . If  $C \in \mathcal{M}_1(\alpha, \beta)$ , the non-trivial component of  $C$  has genus zero by (2) so there is  $\tilde{C} \in \mathcal{M}_1^{\tilde{J}}(\tilde{\alpha}, \tilde{\beta})$  where  $\tilde{\alpha}$  and  $\tilde{\beta}$  are orbit sets of  $\tilde{\lambda}$  and  $\tilde{q}(\tilde{C}) = C$ . If  $\text{int}(\text{supp } \mathcal{R}_{\tilde{\alpha}, \tilde{\beta}}) = (x_1, x_2)$  with  $x_1 \in [0, 1)$ ,

$$0 = a'(x_1)^{\vee} \times \sigma_{\mathcal{R}_{\alpha, \beta, \sigma}}(x_1) = \sum_{n \in \mathbb{Z}} a'(x_1)^{\vee} \times \sigma_{\mathcal{R}_{\tilde{\alpha}, \tilde{\beta}}}(x_1 + n),$$

so by  $q^*$ -positivity of  $\mathcal{R}_{\tilde{\alpha}, \tilde{\beta}}$ ,  $x_2 \leq x_1 + 1$ , proving the claim.

It remains to show: there does not exist a pair  $(\alpha, \beta)$  with  $m^l(\mathcal{R}_{\alpha, \beta, 0}) \leq 1$  and  $\#(\mathcal{M}_1(\alpha, \beta)/\mathbb{R}) = 1$ .

*Case 1.* If there exists  $(\alpha, \beta)$  with  $m^l(\mathcal{R}_{\alpha, \beta, 0}) = 0$  and  $\#(\mathcal{M}_1(\alpha, \beta)/\mathbb{R}) = 1$ , pick one with the minimal  $z(\alpha, \beta) := m(\mathcal{R}_{\alpha, \beta, 0}) + \sum_{x \in [0, 1]} x \cdot m_{\mathcal{P}_\alpha}(x)/N$ . Lemma 4.6 guarantees  $0 \leq x_1 < x_2 < 1$  so that  $[x_1, x_2] \cap \text{supp}(\mathcal{P}_\alpha) = \{x_1, x_2\}$  and  $\varphi_{a'}(x_1) < \varphi_{a'}(x_2)$ . Let  $x_0$  be the largest  $x \in (x_1, x_2)$  with  $\varphi_{a'}(x) = \varphi_{a'}(x_2)$  and let  $\alpha'$  be such that  $\mathcal{R}_{\alpha, \alpha, 0} = \mathcal{T}\mathcal{T}_0$  and  $\mathcal{R}_{\alpha', \alpha, 0} = \mathcal{T}\mathcal{R}_0$  where  $\mathcal{T}_0$  is a local bigon at  $x_2$  and  $\mathcal{R}_0$  is a bigon with  $\text{supp } \mathcal{R}_0 = [x_0, x_2]$  and  $I(\mathcal{R}_0) = 1$ .

Suppose  $\sigma^1, \sigma^2$  and  $\gamma$  satisfy  $\#(\mathcal{M}_1(\alpha', \gamma, \sigma^1)/\mathbb{R}) = \#(\mathcal{M}_1(\gamma, \beta, \sigma^2)/\mathbb{R}) = 1$ . Applying Proposition 4.2(c) on the underlying  $\bar{\mathcal{R}}^1$  and  $\bar{\mathcal{R}}^2$  of  $\mathcal{R}^1 := (\mathcal{P}_{\alpha'}, \mathcal{P}_\gamma, \sigma^1)$  and  $\mathcal{R}^2 := (\mathcal{P}_\gamma, \mathcal{P}_\beta, \sigma^2)$ ,  $\sum m^l(\bar{\mathcal{R}}_0^j) + \sum \bar{I}(\bar{\mathcal{R}}_0^j) = 2$  where  $\bar{\mathcal{R}}_0^j$  is the (unique by (2)) non-local factor of  $\bar{\mathcal{R}}^j$ . If neither  $\bar{\mathcal{R}}^j$  lifts,  $m(\mathcal{P}_\gamma) = m^s(\bar{\mathcal{R}}^1, \bar{\mathcal{R}}^2) \leq 1$ , contradicting Lemma 4.6. Hence,  $\mathcal{R}^2$  does not lift with  $m^l(\mathcal{R}^2) = 0$ , while  $\mathcal{R}_0^1$  lifts to  $(\mathcal{P}_\mathbb{R}^0, \mathcal{P}_\mathbb{R}^1)$  with  $m(\mathcal{P}_\mathbb{R}^1) = m^s(\bar{\mathcal{R}}^1, \bar{\mathcal{R}}^2) = 1$ . By our choice of  $(\alpha, \beta)$ ,  $\mathcal{R}_0^1$  must be a bigon due to the first term in  $z(\alpha, \beta)$ , and  $\gamma = \alpha$  due to the second, contradicting  $\langle \partial^2 \alpha', \beta \rangle = 0$ . Here,  $\alpha, \beta$  and  $\alpha'$  are admissible by  $m^l(\mathcal{R}_{\alpha, \beta, 0}) = 0$  and construction.

*Case 2.* If  $m^l(\mathcal{R}_{\alpha, \beta, 0}) = 1$ , we take any non-local  $\mathcal{R}_{\alpha, \beta, \sigma}$  and show  $\mathcal{M}_1(\alpha, \beta, \sigma) = \emptyset$  for  $(\lambda, J)$  sufficiently close to  $(\pi_{S^1}^* a, \bar{J})$ . Then this holds true for any good  $\lambda$  and generic  $\lambda$ -admissible  $J$  by automatic transversality [18] since  $m^h(\mathcal{P}_\alpha \cup \mathcal{P}_\beta) = 1$ . We proceed similarly to the proof of Proposition 3.11 and only highlight the differences. Suppose a sequence  $C_n \in \mathcal{M}_1^{J_n}(\alpha, \beta, \sigma)$  of  $J_n$ -holomorphic curves with  $(\lambda_n, J_n) \rightarrow (\pi_{S^1}^* a, \bar{J})$  converges to a  $\bar{J}$ -holomorphic building  $\bar{C} = (\bar{C}^1, \dots, \bar{C}^l)$  where  $\bar{C}^i \in \mathcal{M}(\alpha^i, \beta^i, \sigma^i)$  for orbit sets  $\alpha^i$  and  $\beta^i$  of  $\pi_{S^1}^* a$ . Let  $\bar{\mathcal{P}}^{i-1} = \bar{\mathcal{P}}_{\alpha^i}$  and  $\bar{\mathcal{P}}^i = \bar{\mathcal{P}}_{\beta^i}$  for  $1 \leq i \leq l$  and let  $i_1$  and  $i_2$  be the smallest and the largest  $i$  such that  $\bar{\mathcal{R}}^i := (\bar{\mathcal{P}}^{i-1}, \bar{\mathcal{P}}^i, \sigma^i)$  is non-local.

By Lemma 3.10,  $i_1 < i_2$  and by Proposition 4.2(c) on  $(\bar{\mathcal{R}}')^1 = (\bar{\mathcal{P}}^0, \bar{\mathcal{P}}^{i_1}, \sum_{i=1}^{i_1} \sigma^i)$  and  $(\bar{\mathcal{R}}')^2 = (\bar{\mathcal{P}}^{i_1}, \bar{\mathcal{P}}^{i_2}, \sum_{i=i_1+1}^{i_2} \sigma^i)$ ,  $\sum m^l((\bar{\mathcal{R}}')_k^j) + \sum \bar{I}((\bar{\mathcal{R}}')_k^j) = 2$ . Since at least one  $(\bar{\mathcal{R}}')^j$  lifts (see Case 1),  $m^s((\bar{\mathcal{R}}')^1, (\bar{\mathcal{R}}')^2) = 1$  and there are exactly two non-local factors, each with  $\bar{I}((\bar{\mathcal{R}}')_k^j) = 0$ , one of which lifts with  $m^l((\bar{\mathcal{R}}')_k^j) = 2$  and the other does not lift with  $m^l((\bar{\mathcal{R}}')_k^j) = 0$ . In turn,  $\bar{\mathcal{R}}^{i_1} = (\bar{\mathcal{R}}')^1$ ,  $\bar{\mathcal{R}}^{i_2} = (\bar{\mathcal{R}}')^2$  and all other  $\bar{\mathcal{R}}^i$  is local. Write  $m_{\bar{\mathcal{R}}^{i_1}, \bar{\mathcal{R}}^{i_2}}^s = \delta_{x_0}$  and suppose  $\bar{\rho}_{x_0}$  is convex, i.e.,  $m^l(\bar{\mathcal{R}}_0^{i_2}) = 2$  for the non-local factor of  $\bar{\mathcal{R}}^{i_2}$  and  $m^l(\bar{\mathcal{R}}^{i_1}) = 0$ . By Step 2 in the original proof, the non-trivial component  $\bar{C}_0^{i_2}$  of  $\bar{C}^{i_2}$  has a positive end at  $\bar{\rho}_{x_0}(\theta^+)$  and no other end at  $x_0$  while  $\bar{C}^{i_1}$  has a negative end at  $\bar{\rho}_{x_0}(\theta^-)$  and all other ends at some  $\bar{\rho}_x^m(\theta_x^{\max})$ . By Lemma 3.10,

$$\Theta(\bar{C}^{i_1}) = (m(\bar{\mathcal{R}}^{i_1}) - 1)\theta_h/N - \theta^- = 0$$

and

$$\Theta(\bar{C}_0^{i_2}) = (m(\bar{\mathcal{R}}_0^{i_2}) - 2)\theta_h/N + \theta_h + \theta^+ = 0.$$

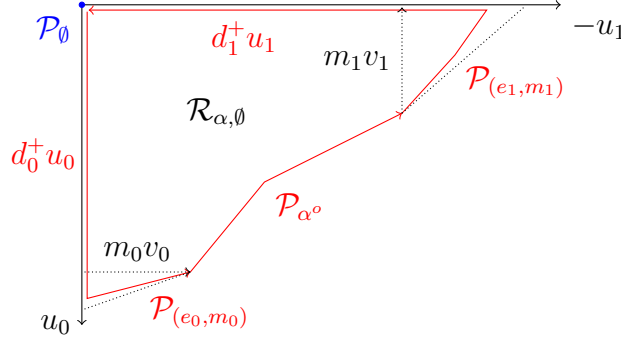


Figure 5:  $\mathcal{R}_{\alpha, \emptyset}$  for orbit sets in  $(S^3, \lambda)$ .

Hence,  $0 < \theta^- < \theta_h$ ,  $-2\theta_h < \theta^+ < -\theta_h$  and there is no flow of  $f_{x_0}$  from  $\theta^-$  to  $\theta^+$ . Draw a similar contradiction if  $\bar{\rho}_{x_0}$  is concave.  $\square$

## 4.2 Toric contact $L(p, q)$ , $p \neq 0$

Let  $a : \mathbb{I} \rightarrow \mathfrak{t}^2$  be an orbital moment map and suppose  $a(i)^\vee \sim (-1)^i u_i$  for a primitive  $u_i \in \Lambda$  for  $i = 0, 1$ . Collapse  $u_i$ -orbits at each  $\{i\} \times T^2$  to obtain a contact lens space  $(Y, \bar{\lambda})$  [14]. If  $(u_0|u_1) \sim \begin{pmatrix} p & 0 \\ q & 1 \end{pmatrix}$  up to  $SL_2(\mathbb{Z})$ ,  $Y$  is diffeomorphic to  $L(p, q)$  with  $H_1(Y) = \Lambda / \text{span } u_i$ . Fix  $v_i \in \mathbb{Z}^2$  so that  $\det(u_i|v_i) = (-1)^i$ . Over each new embedded orbit  $e_i$  with image  $(\{i\} \times T^2)/u_i$ ,  $v_i$ -action trivializes  $\xi$ , with respect to which  $e_i$  is elliptic with rotation angle  $\phi_i$  given by  $a'(i)^\vee \sim v_i - \phi_i u_i$ . Trivialize  $\xi$  over orbits in  $Y^o = \text{int}(\mathbb{I}) \times T^2$  as before.

To perturb  $\bar{\lambda}$ , let  $v_a(x)$  be as in §2.2 on  $\text{int}(\mathbb{I})$  and  $v_a(i) = v_i$  for  $i = 0, 1$ . Define  $\mathcal{A}_a$ ,  $\Xi_L$  and  $N$  as before and choose disjoint neighbourhoods  $U_x$  of  $x \in \Xi_L$  on which  $a' \times a''$  does not vanish. We take a small perturbation  $\lambda$  of  $\bar{\lambda}$  which is good on each  $U_x$  for  $x \in \Xi_L \setminus \{0, 1\}$  and unperturbed elsewhere. Assume  $m\phi_i \notin \mathbb{Z}$  for  $0 < m < N$  and fix a generic  $\lambda$ -admissible almost complex structure  $J$  on  $\mathbb{R} \times Y$ .

Additionally, fix  $\epsilon_0, \epsilon_1 > 0$ ,  $\tilde{x}_0 < -\epsilon_0$  and  $\tilde{x}_1 > 1 + \epsilon_1$  and let  $V_0 := [\tilde{x}_0, -\epsilon_0]$ ,  $V_1 := (1 + \epsilon_1, \tilde{x}_1]$ ,  $V'_0 := [\tilde{x}_0, 0]$  and  $V'_1 := [1, \tilde{x}_1]$ . Extend  $a$  to an orbital moment map  $\tilde{a}$  on  $[\tilde{x}_0, \tilde{x}_1]$  so that, for each  $i = 0, 1$ , (i)  $\tilde{a}'|_{V'_i \setminus V_i}$  does not annihilate any  $nv_i - n'u_i \in \text{span}_{\mathbb{Z}}\{v_i, u_i\}$  with  $|n| < N$ ; (ii)  $\tilde{a}$  is convex on  $V_i$ ; and (iii)  $a(i) \times \tilde{a}'(x) \geq 0$  for  $x \in V_i$  with equality only at  $x = \tilde{x}_i$ . In this section, paths will be functions on  $[\tilde{x}_0, \tilde{x}_1]$ .

**Definition 4.7.** Let  $\alpha$  and  $\beta$  be orbit sets of  $\lambda$  with  $[\alpha] = [\beta] \in H_1(Y) \cong \mathbb{Z}/p$ .

- (a) For  $i = 0, 1$ , let  $w_{i,n} := nv_i - \lfloor n\phi_i \rfloor u_i$ . To  $\gamma = (e_i, m_i)$ , we associate the unique  $\tilde{a}$ -compatible path  $\mathcal{P}_\gamma$  with  $m_{\mathcal{P}_\gamma}^h \equiv 0$  and  $m_{\mathcal{P}_\gamma}^e := \sum_n b_n \delta_{x_n}$ , where  $n$  appears in  $p_{\phi_i}^+(m_i)$  with multiplicity  $b_n$  and  $x_n$  is the unique  $x \in V_i$  with  $\tilde{a}'(x)^\vee \sim w_{i,n}$ .



To an orbit set  $\gamma^o$  in  $Y^o$ , *associate*  $\mathcal{P}_{\gamma^o}$  as before. In general, we *associate* to  $\gamma = \gamma^o \cup \{(e_0, m_0), (e_1, m_1)\}$  the path  $\mathcal{P}_\gamma := \mathcal{P}_{(e_0, m_0)} \mathcal{P}_{\gamma^o} \mathcal{P}_{(e_1, m_1)}$ .

- (b) Write  $[\mathcal{P}_\beta] - [\mathcal{P}_\alpha] = d_0 u_0 + d_1 u_1$  and  $d_i = d_i^+ - d_i^-$  such that  $d_i^\pm \geq 0$  and  $d_i^+ d_i^- = 0$  for each  $i$ . To  $\alpha$  and  $\beta$ , we *associate* the region  $\mathcal{R}_{\alpha, \beta} = (\mathcal{P}_0^+ \mathcal{P}_\alpha \mathcal{P}_1^+, \mathcal{P}_0^- \mathcal{P}_\beta \mathcal{P}_1^-)$  where  $\mathcal{P}_i^\pm$  is  $(u_i, 1, d_i^\pm, 0)$  at  $\tilde{x}_i$  and vanishes elsewhere.

Note  $\mathcal{P}_{(e_i, m_i)}$  is an interpretation of  $\Lambda_{\phi_i}^+(m_i)$  in §2.1 as a path. Figure 5 shows  $\mathcal{R}_{\alpha, \emptyset} = (\mathcal{P}^+, \mathcal{P}_\emptyset)$  for an orbit set  $\alpha$  of  $(S^3, \lambda)$ , where  $\mathcal{P}^+$  goes around clockwise.

**Theorem 4.8.** Let  $(Y, \lambda, J)$  as above. For admissible orbit sets  $\alpha$  and  $\beta$  of  $\lambda$ ,  $\langle \partial\alpha, \beta \rangle = 1$  if and only if  $\mathcal{R}_{\alpha, \beta} = \mathcal{T}_1 \mathcal{R}' \mathcal{T}_2$  where  $\mathcal{T}_i$  are trivial and  $\mathcal{R}'$  is non-local, indecomposable,  $\tilde{a}$ -positive, minimally positive and almost minimally decorated.

We observe the following property of  $\mathcal{R}_{\alpha, \beta}$  on  $V_i$ :

**Lemma 4.9.** If the function  $(\tilde{a}')^\vee \times \sigma_{\mathcal{R}_{\alpha, \beta}}$  vanishes at  $x_i \in V'_i$  for  $i = 0$  (or 1), then  $x_i \in V_i$  and  $\sigma_{\mathcal{R}_{\alpha, \beta}} = 0$  for  $x < x_0$  (or  $x > x_1$ ). In particular, if it is positive at  $x = i$ , then it is non-negative on  $V'_i$ ,  $\mathcal{R}_{\alpha, \beta}$  is not minimally decorated and  $\sum_{x \in V_i} I_{\mathcal{R}_{\alpha, \beta}}(x) > 0$ .

*Proof.* The first statement follows from the definition of  $\tilde{a}$  and  $\Lambda_{\phi_i}^+$ . The rest follows from this and that all edges of  $\mathcal{R}_{\alpha, \beta}$  are elliptic convex.  $\square$

**Proposition 4.10.** To adapt previous results, write  $\alpha = \alpha^o \cup \{(e_0, m_0^+), (e_1, m_1^+)\}$ ,  $\beta = \beta^o \cup \{(e_0, m_0^-), (e_1, m_1^-)\}$  and  $(-1)^i \sigma_{\mathcal{R}_{\alpha, \beta}}(i) = (m_i^- - m_i^+) v_i - c_i u_i$ .

- (a) Proposition 3.2 still holds and  $c_\tau(Z) = c_0 + c_1$ .
- (b) Lemma 3.5 holds after replacing  $a$  with  $\tilde{a}$  in the statement.
- (c) Proposition 3.6 now asserts:  $\alpha'$  and  $\beta'$  do not share orbits;  $C'$  has at most one end at covers of  $e_0$  or  $e_1$ ;  $g(C') = 0$ ;  $\mathcal{R}_{\alpha, \beta} = \mathcal{T}_1 \mathcal{R}' \mathcal{T}_2$  where  $\mathcal{R}'$  is indecomposable ( $\mathcal{R}'$  may differ from  $\mathcal{R}_{\alpha', \beta'}$ ) and  $\mathcal{T}_i$  are trivial; and  $\mathcal{M}_1(\alpha, \beta) \cong \mathcal{M}_1(\alpha', \beta')$ .
- (d) We say  $a : \mathbb{I} \rightarrow (\mathfrak{t}^2)^*$  is  $\mathcal{R}$ -adapted if it admits an  $\mathcal{R}$ -adapted extension  $\tilde{a}$ . Proposition 3.14 holds with this definition.

*Proof.* (a) Let  $\pi$  and  $\psi$  be as in the original proof. Define  $G^o$  by (8) for  $\alpha^o$  and  $\beta^o$ ,

$$G_0 := \{(s, \epsilon(1 - |s|))\}_{s \in [-1, 1]}, \quad G_1 := \{(s, 1 - \epsilon(1 - |s|))\}_{s \in [-1, 1]},$$

and let  $V$  be the set of multivalent vertices of  $G_0 \cup G^o \cup G_1$ . Since  $H_2(Y) = 0$ , we compute  $I(\alpha, \beta, Z)$  on a smooth surface  $S$  as in §2.1 subject to: (i)  $\pi(S) \subset G_0 \cup G^o \cup G_1 \cup B_{\epsilon/2}(V)$ ; (ii) for each component  $E$  of  $G \setminus B_{\epsilon/2}(V)$ ,  $\pi_S^{-1}(E)$  consists

of  $|c_i|$  disjoint embedded  $u_i$ -invariant disks if  $(i, 0) \in E$  for  $i = 0, 1$  and as before, otherwise. We can construct such an  $S$  by gluing these pieces as before.

We are ready to compute  $I(\alpha, \beta, [S])$ . It suffices to compare  $\sum_{x \in V_i} I_{\mathcal{R}_{\alpha, \beta}}(x)$  to

$$I_i := \#(\zeta^{-1}(0) \cap S_i) + \#(S_i \cap S'_i) + CZ^I((e_i, m_i^+)) - CZ^I((e_i, m_i^-))$$

where  $\zeta := x(1-x)\partial_x \in \Gamma(\xi)$  and  $S'_i := (\psi \times \text{id}_Y)(S_i)$ . The first two terms are equal to  $c_i$  and  $c_i(m_i^+ + m_i^-)$ , respectively, while  $CZ^I((e_i, m_i^\pm)) - [m_i^\pm \phi_i] - m(\mathcal{P}_{(e_i, m_i^\pm)})$  is twice the area  $A_i^\pm$  under the graph of  $\Lambda_{\phi_i}^+(m_i^\pm)$  by Pick's theorem. Using these and  $d_i = c_i + [m_i^+ \phi_i] - [m_i^- \phi_i]$ , we get

$$I_i = (2A_i^+ - 2A_i^- + c_i(m_i^+ + m_i^-)) + (m(\mathcal{P}_{(e_i, m_i^+)}) - m(\mathcal{P}_{(e_i, m_i^-)}) + d_i).$$

The first summand equals  $\sum_{x \in V_i} Q_{\mathcal{R}_{\alpha, \beta}}(x)$  and the second equals  $\sum_{x \in V_i} CZ_{\mathcal{R}_{\alpha, \beta}}(x)$ .

(b) Argue as before on  $\mathbb{I}$  and use Lemma 4.9 on  $V'_i$ .

(c) By Lemma 3.5 and by symmetry, it suffices to assume  $\sigma_{\mathcal{R}_{\alpha, \beta}}(0) \neq 0$ . By Lemma 4.9,  $\sigma_{\mathcal{R}_{\alpha, \beta}}(1) = 0$ . If  $C$  is irreducible,

$$\sum CZ_\tau(e_0^{n_i^+}) = \sum (2[n_i^+ \phi_0] + 1) = p^+ + 2[m_0^+ \phi_0]$$

and

$$\sum CZ_\tau(e_0^{n_i^-}) = \sum (2[n_i^- \phi_-] - 1) = -p^- + 2[m_0^- \phi_0]$$

where each sum is over the entries of the partition  $(n_1^\pm, \dots, n_{p^\pm}^\pm)$  of  $m_0^\pm$  given by  $C$ . Substituting these into (2) and using  $d_0 = c_0 + [m_0^+ \phi_0] - [m_0^- \phi_0]$ ,

$$1 = \text{ind}(C) = 2(g(C) - 1 + d_0 + p^+ + \max\{0, p^- - 1\}) + \sum (\pm CZ(\rho_j^\pm) + 1)$$

where the sum is over positive/negative ends of  $C$  at  $\rho_j^\pm$  in  $Y^o$ . Thus, if  $p^\pm > 0$ ,  $p^+ = p^- = 1$  and  $d_0 = 0$ . By a simple fact for special partitions (§2.1),  $p_{\phi_0}^- = (m_0^-)$  implies  $1 \in p_{\phi_0}^+(m_0^-)$ , i.e.  $m_{\mathcal{P}_\beta}(x_0) > 0$  where  $v_a(x_0) = v_0 - [\phi_0]u_0$ . Since  $\mathcal{R}_{\alpha, \beta}$  is positive with  $d_0 = 0$ ,  $m_{\mathcal{P}_\alpha}(x_0) > 0$  and by index,  $m_{\mathcal{P}_\alpha}|_{V_0} = \delta_{x_0}$ . Then, by convexity of  $\tilde{a}$  on  $V_0$ ,  $\mathcal{R}_{\alpha, \beta}$  violates  $\tilde{a}$ -positivity at 0. Therefore,  $p^+p^- = 0$ . Moreover,  $d_0 + p^+ > 0$  by  $\tilde{a}$ -positivity, so  $d_0 + p^+ = 1$ ,  $p^- \in \{0, 1\}$ ,  $g(C) = 0$  and  $\sum_{x \in \mathbb{I}} m_{\mathcal{R}_{\alpha, \beta}}^l(x) = 1$ . By the last condition,  $\mathcal{R}_{\alpha, \beta}$  cannot decompose at  $x \in \mathbb{I}$  and  $\alpha$  and  $\beta$  do not share orbits in  $Y^o$  either. The rest follows easily from this and Lemma 4.9.

(d) Assume  $\sigma_{\mathcal{R}}(1) = 0$  as above. Let  $\{\tilde{a}_r\}_{r \in [0, 1]}$  be a path given by Lemma 3.13 except: if  $m_0^+ + m_0^- > 0$  (so  $\text{supp } m_{\mathcal{R}}|_{V_0} = \{x_1 < \dots < x_{n_0}\} \neq \emptyset$  and  $x_{n_0} > \tilde{x}_0$ ), we additionally require  $\tilde{b}$  to be smooth on  $[x_{n_0}, z_{n_0})$  for  $z_{n_0} > 0$ , which is possible since

$a(0) \times v_{\mathcal{R}}(x_{n_0}) > 0$ . Let  $\bar{\lambda}_r$  be the contact form on  $Y$  obtained for each  $a_r := \tilde{a}_r|_{\mathbb{I}}$  as above and  $\phi_0^r$  the return angle of the orbit of  $\bar{\lambda}_r$  at  $x = 0$ .

Since  $\Lambda_{\phi}^+(m) = \Lambda_{\phi'}^+(m)$  implies  $\Lambda_{\phi}^+(m') = \Lambda_{\phi'}^+(m')$  for all  $m' \leq m$ , we may assume that  $\alpha$  and  $\beta$  do not share orbits by (c) and that, for  $0 < m \leq \max\{m_0^+, m_0^-\}$ ,  $m\phi_0^r$  never crosses an integer during the deformation by the above requirement. This guarantees orbit sets  $\alpha_r$  and  $\beta_r$  of  $\lambda_r$  with  $\mathcal{R}_{\alpha_r, \beta_r} = \mathcal{R}$  as well as non-degeneracy of the orbits involved. We can carry out the rest of the original proof with minor adjustments.  $\square$

*Proof of Theorem 4.8.* By Lemma 4.9, Proposition 4.10 and §3, it suffices to show  $\langle \partial\alpha, \beta \rangle = 1$  when  $I(\mathcal{R}_{\alpha, \beta}) = 1$ ,  $\mathcal{R}_{\alpha, \beta}$  is indecomposable,  $\sigma_{\mathcal{R}_{\alpha, \beta}}(0) \neq 0$  and  $\alpha$  and  $\beta$  do not share  $e_1$ . We can also reduce  $\sum_{x \in \mathbb{I}} m_{\mathcal{R}_{\alpha, \beta}}(x)$  to 1 by induction as in the proof of Proposition 3.16 and assume  $m_{\mathcal{P}_{\beta}}|_{\mathbb{I}} \equiv 0$  by duality as in Proposition 3.15(iii). Hence,  $C \in \mathcal{M}_1(\alpha, \beta)$  has one positive end at  $\check{h}_{x^o}$  for  $x^o \in \text{int}(\mathbb{I})$  and the only other end is: (i) none, (ii) a negative end at  $e_0^{m_0^-}$ , or (iii) a positive end at  $e_0^{m_0^+}$ . Assume  $u_0 = (1, 0)$  and  $1 < \phi_0 < 2$  by  $SL_2(\mathbb{Z})$ -symmetry.

If  $|m_0^{\pm}| \leq 1$ , Proposition 4.10(d) allows us to deform  $\bar{\lambda}$  to the pullback of  $\alpha$  under a suitable identification of  $Y' := \pi_{\mathbb{I}}^{-1}([0, 1 - \epsilon])$  (diffeomorphic to  $D^2 \times S^1$ ) with a subset of  $(S^2 \times S^1, \alpha)$  in  $I_C = \aleph_C + 1$  case of [17, Theorem A.1]. In each of our three cases, a Morse-Bott argument [1] gives a unique member of  $\mathcal{M}_1(\alpha, \beta)/\mathbb{R}$  from an  $(\mathbb{R} \times S^1)$ -family of  $J$ -holomorphic curves there: use (a1) for case (i); (a2) with  $p = 1, p' = 2$  for (ii); and (a3) with  $p = p' = 1$  for (iii).

Otherwise, define  $\pi : \mathbb{R} \times Y' \rightarrow S^1$  and  $q : \mathbb{R} \times Y' \rightarrow \mathbb{R} \times Y'$  by  $\pi(s, x, t_1, t_2) = t_2$  and  $q(s, x, t_1, t_2) = (s, x, t_1, mt_2)$ . A  $J$ -holomorphic cylinder  $u : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times Y'$  with  $\deg(\pi \circ u) = \pm m$  lifts (in  $m$  different ways) to a  $\tilde{J}$ -holomorphic cylinder  $\tilde{u}$  where  $q_*\tilde{J} = Jq_*$ . Here,  $\tilde{J}$  is  $\tilde{\lambda}$ -admissible for a perturbation of  $\bar{\lambda}$  using  $f_{x^o}$  with  $2m$  critical points but we can pick one  $\tilde{u}$  (with an end at  $\bar{\rho}_{x^o}(\theta_0)$  for a local minimum  $\theta_0$ ) and deform away any other local minimum  $\theta$  of  $f_{x^o}$  since  $\tilde{u}$  stays away from  $\mathbb{R} \times \bar{\rho}_{x^o}(\theta)$ . By  $T^2$ -action and Proposition 4.10(d), we reduce the above case.  $\square$

### 4.3 Toric contact $S^1 \times S^2$

The discussions from §4.2 work here except, to account for  $u_0 = \pm u_1$ :

**Definition 4.11.** The region  $\mathcal{R}_{\alpha, \beta, d}$  associated to  $\alpha, \beta$  and  $d \in \mathbb{Z}$  is the pair  $(\mathcal{P}_0^+ \mathcal{P}_{\alpha} \mathcal{P}_1^+, \mathcal{P}_0^- \mathcal{P}_{\beta} \mathcal{P}_1^-)$  as in Definition 4.7 except we impose  $d_0 = d$ . (Note  $d_0$  and  $d_1$  are not uniquely determined otherwise.)

**Theorem 4.12.** Define  $(\lambda, J)$  on  $Y$  as in §4.2. For admissible orbit sets  $\alpha$  and  $\beta$  of  $\lambda$ ,  $\langle \partial\alpha, \beta \rangle = 1$  if and only if there exists a unique  $d$  such that  $\mathcal{R}_{\alpha, \beta, d} = \mathcal{T}_1 \mathcal{R}' \mathcal{T}_2$  where  $\mathcal{T}_i$  are trivial and  $\mathcal{R}'$  is non-local, indecomposable,  $\tilde{a}$ -positive, minimally positive and almost minimally decorated.

**Remark 4.13.** (Uniqueness of  $d$ ) If  $\mathcal{R}_{\alpha,\beta,d}$  is minimally positive,  $d = 0$  or  $1$ . Suppose both  $\mathcal{R}_{\alpha,\beta,0}$  and  $\mathcal{R}_{\alpha,\beta,1}$  satisfy the criteria in Theorem 4.12 and write  $\alpha = \alpha' \cup \gamma$  and  $\beta = \beta' \cup \gamma$ . By an analogue of Lemma 4.9,  $\text{int}(\text{supp } \mathcal{R}_{\alpha,\beta,1}) = (\tilde{x}_0, x^o)$  and  $\text{int}(\text{supp } \mathcal{R}_{\alpha,\beta,0}) = (x^o, \tilde{x}_1)$  where  $m_{\mathcal{R}_{\alpha',\beta',0}}^l = \delta_{x^o}$  and as in Remark 4.5,  $\text{supp}(m_{\mathcal{P}_{\alpha \cup \beta}}) = \{x^o\}$ . Therefore,  $(a')^\vee \times u_0$  vanishes precisely at  $x^o$ ,  $a'$  is convex at  $x^o$  (by  $a \times a' > 0$ ),  $\alpha' = \check{h}_{x^o}$ ,  $\beta' = \emptyset$  and  $\gamma$  consists of orbits at  $x^o$ .

*Proof of Theorem 4.12.* By Lemma 4.9, we may re-use arguments from §4.2. In case of non-unique  $d$ , the two non-zero contributions to  $\langle \partial\alpha, \beta \rangle$  cancel.  $\square$

#### 4.4 Map from $ECC(L(p, q))$ to $ECC(T^3)$

Consider an orbital moment map  $a_T : \mathbb{R}/2\mathbb{Z} \rightarrow (\mathfrak{t}^2)^*$  and suppose there are  $\tilde{x}_0 \in (-1/2, 0)$  and  $\tilde{x}_1 \in (1, 3/2)$  so that  $a_L := a_T|_{\mathbb{I}}$  and  $\tilde{a}_L := a_T|_{[\tilde{x}_0, \tilde{x}_1]}$  satisfy the conditions of  $a$  and  $\tilde{a}$  in §4.2. Suppose further that, for each  $i = 0, 1$ ,  $a_L(i) \times a'_T$  is positive on  $\mathbb{I}$  and negative on  $(\tilde{x}_1, \tilde{x}_0 + 2)$ . (In particular  $p \neq 0$ .) As in §4.1 and §4.2, choose a good perturbation  $\lambda_T$  of  $\pi_{\mathbb{R}/2\mathbb{Z}}^* a_T$  on  $(\mathbb{R}/2\mathbb{Z}) \times T^2$  and  $\lambda_L$  of  $\tilde{a}_L$  on  $L(p, q)$ , as well as generic  $\lambda_T$  and  $\lambda_L$ -admissible  $J_T$  and  $J_L$ . Then, for any orbit set  $\alpha$  of  $\lambda_L$  with  $[\alpha] = 0$ ,  $\mathcal{R}_{\alpha,\emptyset} = (\mathcal{P}^+, 0)$  for a unique  $\mathcal{P}^+ : [\tilde{x}_0, \tilde{x}_1] \rightarrow \mathcal{V}$ .

**Proposition 4.14.** Define  $\Phi : ECC(L(p, q), \lambda_L, J_L, 0) \rightarrow ECC(T^3, \lambda_T, J_T, 0)$  by  $\mathcal{R}_{\alpha,\emptyset} = (\mathcal{P}_{\Phi(\alpha)}, 0)$ . Then,  $I(\alpha, \emptyset) = I(\Phi(\alpha), \emptyset)$  and  $\partial_T \Phi = \Phi \partial_L$ .

*Proof.* Let  $\mathcal{R} = \mathcal{R}_{\Phi(\alpha),\beta,\sigma}$  for any  $\beta$  and  $\sigma$  with  $\#(\mathcal{M}_1^{J_T}(\Phi(\alpha), \beta, \sigma)/\mathbb{R}) = 1$ . Suppose  $\sigma_{\mathcal{R}}(x) \neq 0$  for  $x \in (\tilde{x}_1, \tilde{x}_0 + 2)$ . By positivity of  $\mathcal{R}$  at  $\tilde{x}_0 + 2$  and the condition on  $a'_T$  on  $(\tilde{x}_1, \tilde{x}_0 + 2)$ ,  $\text{supp}(\mathcal{R})$  contains  $[\tilde{x}_1, x]$ , and similarly  $[x, \tilde{x}_0 + 2]$ . Moreover, since  $I(\mathcal{R}) = 1$ ,  $\text{supp } \mathcal{R}$  contains  $V = [1, \tilde{x}_0 + 2]$  or  $[\tilde{x}_1, 2]$ . Then  $\mathcal{P}_{\alpha}|_V \equiv 0$  contradicting Lemma 4.6. Hence,  $\text{supp } \mathcal{R} \subset [\tilde{x}_0, \tilde{x}_1]$  and  $\mathcal{P}_{\beta} = \mathcal{P}_{\tilde{x}_0} \mathcal{P}_0 \mathcal{P}_{\beta^o} \mathcal{P}_1 \mathcal{P}_{\tilde{x}_1}$  with  $\text{supp } \mathcal{P}_{\tilde{x}_i} = \{\tilde{x}_i\}$ ,  $\text{supp } \mathcal{P}_i \subset \text{int}(V_i)$  and  $\text{supp } \mathcal{P}_{\beta^o} \subset \mathbb{I}$ . If  $[\mathcal{P}_i] = m_i v_i - n_i u_i$ , then  $n_i \leq \phi_i m_i$  by convexity of  $\tilde{a}_L|_{V_i}$  and  $n_i > \phi_i m_i - 1$  by  $I(\mathcal{R}) = 1$ . Thus, by Definition 4.7,  $[\mathcal{P}_{\Phi(\beta')}] = [\mathcal{P}_{\beta}]$  and  $(\mathcal{P}_{\Phi(\beta')}, \mathcal{P}_{\beta}, 0)$  is positive for the orbit set  $\beta' := \{(e_0, m_0), (e_1, m_1)\} \cup \beta^o$  of  $\lambda_L$ . Since  $(\mathcal{P}_{\Phi(\alpha)}, \mathcal{P}_{\Phi(\beta')}, \sigma)$  is positive by Lemma 4.9 and  $\bar{I}(\mathcal{R}) = 0$ ,  $\beta = \Phi(\beta')$  by Lemma 3.9. Finally, if  $m(\mathcal{P}_{\Phi(\alpha)}) = 2$  and  $\mathcal{P}_{\beta} = 0$ , then  $\mathcal{P}_{\Phi(\alpha)}|_{\mathbb{I}} \equiv 0$ , contradicting  $I(\mathcal{R}) = 1$ . Hence,  $(\mathcal{P}_{\Phi(\alpha)}, \mathcal{P}_{\beta})$  has a unique relevant lift, namely  $\mathcal{R}_{\alpha\beta'}$ , and the result follows from Theorem 4.4 and Theorem 4.8.  $\square$

If  $a_T(0) \sim (1, 0)$ ,  $a_T(1) \sim (0, 1)$  and  $a_T$  is convex everywhere, we get:

**Corollary 4.15.** [8, Conjecture A.3] If  $(S^3, \bar{\lambda}_L)$  is the boundary of a convex toric domain,  $\Phi$  as above is a chain map.

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